Approximate tensorization of the relative entropy for noncommuting conditional expectations

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Technical University of Munich

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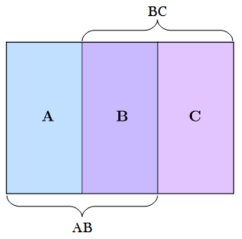

Relative entropy: $D(\rho \| \sigma) := \operatorname{tr}[\rho(\log \rho - \log \sigma)]$

Approximate tensorization of the relative entropy

Given $\Lambda = ABC$, it is an inequality of the form:

$$D(\rho_\Lambda \| \sigma_\Lambda) \leq c \left[D_{AB}(\rho_\Lambda \| \sigma_\Lambda) + D_{BC}(\rho_\Lambda \| \sigma_\Lambda) \right] + d \, ,$$

for $\rho_\Lambda, \sigma_\Lambda \in \mathcal{D}(\mathcal{H}_{ABC})$, constants $c \geq 1$ and $d \geq 0$, and for $D_X(\rho_\Lambda \| \sigma_\Lambda)$ a suitable *conditional relative entropy* in $X \subset \Lambda$.

Motivation

- **Modified logarithmic Sobolev inequalities**

Given a quantum Markov semigroup $\left\{ \mathrm{e}^{t\mathcal{L}} \right\}_{t \geq 0}$ and denoting $\rho_t := \mathrm{e}^{t\mathcal{L}}(\rho)$, a *modified logarithmic Sobolev constant* yields an inequality of the form:

$$D(\rho_t \| \sigma) \leq \mathrm{e}^{-t\alpha(\mathcal{L})} \, D(\rho_0 \| \sigma) \, ,$$

for σ such that $\mathcal{L}(\rho) = 0$.

Classical spin systems: The key ingredient in modern proofs of MLSI constants is a result of *quasi-factorization* or *approximate tensorization* of the entropy.

Quantum systems: Can we do something similar?

- **Generalization of strong subadditivity**

Given $\rho_{ABC} \in \mathcal{D}(\mathcal{H}_{ABC})$, the strong subadditivity (SSA) inequality is:

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}) \, ,$$

where $S(\rho) := -\operatorname{tr}[\rho \log \rho]$ is the von Neumann entropy.

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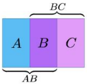

Quasi-factorization / Approximate tensorization of the relative entropy $\Lambda = ABC$

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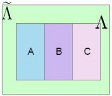

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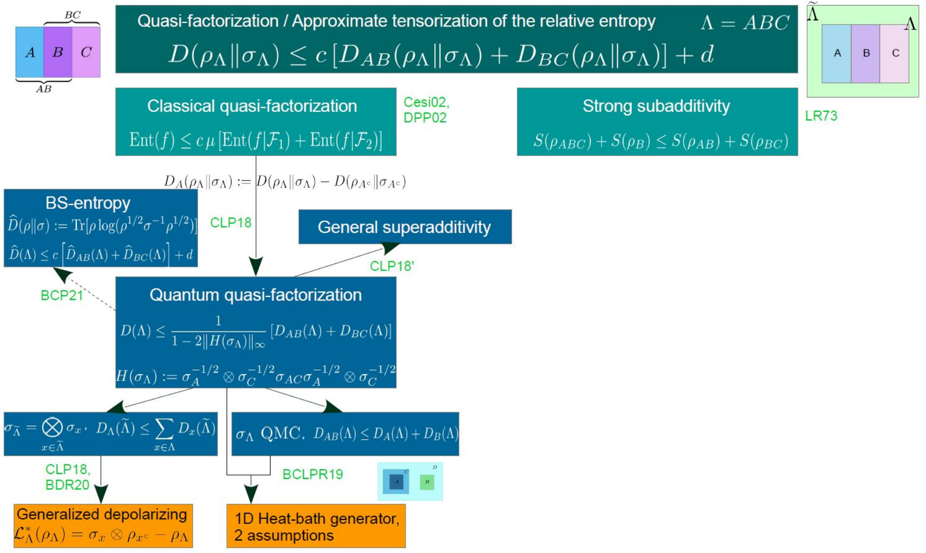
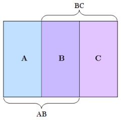


Quasi-factorization for the relative entropy



$$D_A(\rho_{ABC}||\sigma_{ABC}) := D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_{BC}||\sigma_{BC})$$

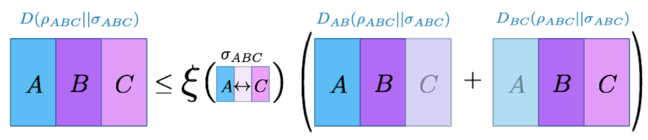
Quasi-factorization for the CRE (C.-Lucia-Pérez García '18)

Let $\mathcal{H}_{ABC}$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. The following holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{AC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where

$$\xi(\sigma_{AC}) = \frac{1}{1 - 2\left\| \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \, \sigma_{AC} \, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC} \right\|_\infty}.$$



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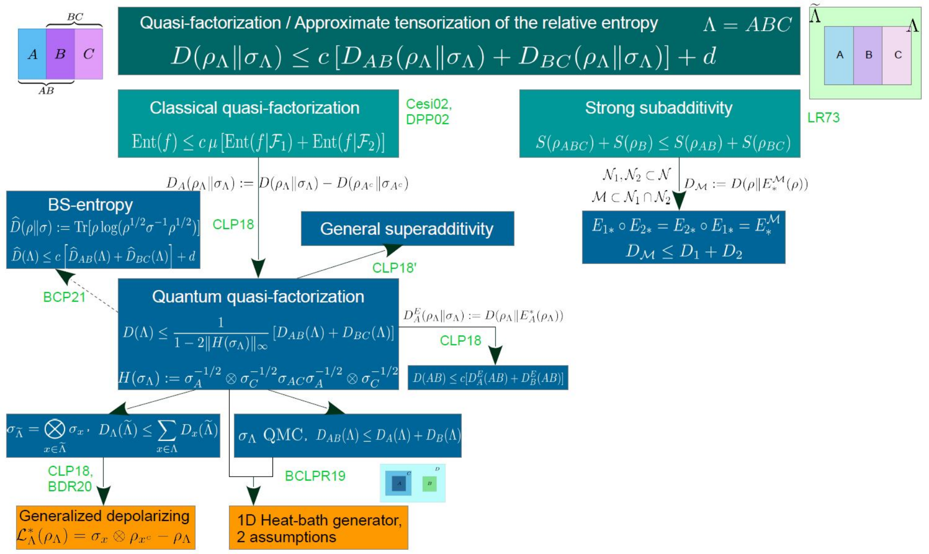

Conditional expectations

Conditional expectations

Let $\mathcal{M} \subset \mathcal{N}$ and $\sigma \in \mathcal{D}(\mathcal{M})$. A linear map $E : \mathcal{N} \to \mathcal{M}$ is a *conditional expectation* with respect to σ of $\mathcal{N}$ onto $\mathcal{M}$ if:

- For all $X \in \mathcal{N}$, $\|E(X)\| \leq \|X\|$.
- For all $X \in \mathcal{M}$, $E(X) = X$.
- For all $X \in \mathcal{N}$, $\operatorname{tr}[\sigma E(X)] = \operatorname{tr}[\sigma X]$.

Generalization of strong subadditivity

In terms of the relative entropy, the **strong subadditivity of entropy** (Lieb-Ruskai '73) takes the form

$$D\left(\rho_{ABC} \Big\| \rho_B \otimes \frac{\mathbb{1}_{AC}}{d_{\mathcal{H}_{AC}}}\right) \leq D\left(\rho_{ABC} \Big\| \rho_{AB} \otimes \frac{\mathbb{1}_{C}}{d_{\mathcal{H}_{C}}}\right) + D\left(\rho_{ABC} \Big\| \rho_{BC} \otimes \frac{\mathbb{1}_{A}}{d_{\mathcal{H}_{A}}}\right) .$$

For $\mathcal{M} \subset \mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{N}$, if $E^{\mathcal{M}}, E_1, E_2$ are the conditional expectations onto $\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2$, respectively, we have

$$D(\rho \| E^{\mathcal{M}}_*(\rho)) \leq D(\rho \| E_{1*}(\rho)) + D(\rho \| E_{2*}(\rho)) \Leftrightarrow E_{1*} \circ E_{2*} = E_{2*} \circ E_{1*} = E^{\mathcal{M}}_* .$$

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$$D(\rho\|E_*^{\mathcal{M}}(\rho))\leq D(\rho\|E_{1*}(\rho))+D(\rho\|E_{2*}(\rho))\Leftrightarrow E_{1*}\circ E_{2*}=E_{2*}\circ E_{1*}=E_*^{\mathcal{M}}\,.$$

Define $E_{A*}:=\lim\limits_{t\to\infty}\mathrm{e}^{t\mathcal{L}_A^*}$. Then,

$$D(\rho\|E_{A\cup B*}(\rho))\leq D(\rho\|E_{A*}(\rho))+D(\rho\|E_{B*}(\rho))\Leftrightarrow E_{A*}\circ E_{B*}=E_{B*}\circ E_{A*}=E_{A\cup B*}\,.$$

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In general, we present conditions in (Bardet-C.-Rouzé '21) for which

$$D(\rho \| E_{A \cup B *}(\rho)) \leq c \left[D(\rho \| E_{A*}(\rho)) + D(\rho \| E_{B*}(\rho)) \right] + d$$

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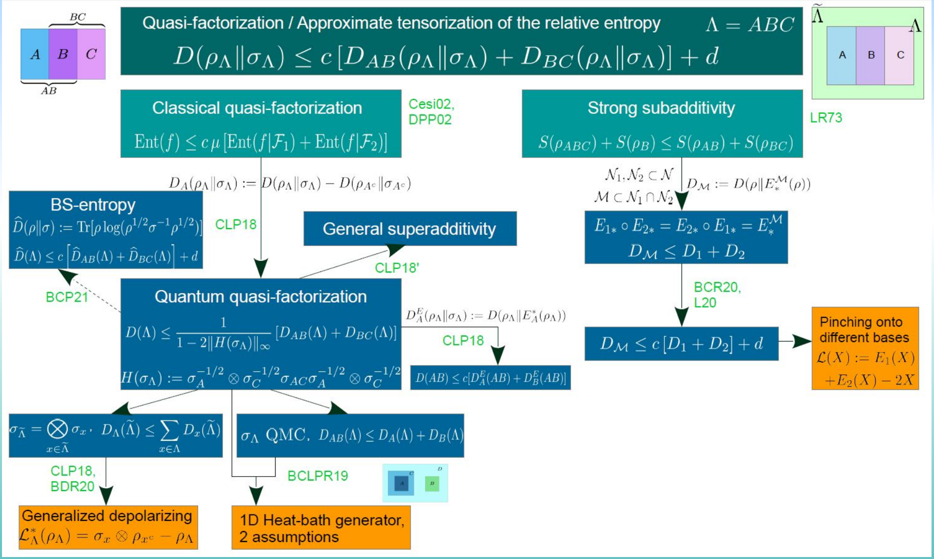
Quasi-factorization / Approximate tensorization



Main results

Take $\mathcal{M} \subset \mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{N}$ and $E^{\mathcal{M}}, E_1, E_2$ the conditional expectations onto $\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2$, respectively.

For **classical systems**, these inequalities take the form

$$D(\rho \| E_*^{\mathcal{M}}(\rho)) \leq \frac{1}{1-2c_1} \left[D(\rho \| E_{1*}(\rho)) + D(\rho \| E_{2*}(\rho)) \right] ,$$

where $\sigma = E_*^{\mathcal{M}}(\sigma)$ and

$$c_1 := \left\| E_1 \circ E_2 - E^{\mathcal{M}} : \mathbb{L}_1(\sigma) \to \mathbb{L}_\infty(\mathcal{N}) \right\| .$$

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$$D(\rho \| E_*^{\mathcal{M}}(\rho)) \leq c \left[D(\rho \| E_{1*}(\rho)) + D(\rho \| E_{2*}(\rho)) \right] + d \, ,$$

where:

- d measures the correction from the classical case.
- c is a finite-temperature relaxation from $c = 1$.

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Approximate Tensorization under change of measure

Consider $E^{(0),\mathcal{M}}, E_1^{(0)}, E_2^{(0)}$ the doubly stochastic conditional expectations onto $\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2$, respectively. Assume that:

$$D(\rho \| E_*^{(0),\mathcal{M}}(\rho)) \leq D(\rho \| E_{1*}^{(0)}(\rho)) + D(\rho \| E_{2*}^{(0)}(\rho)) + d\,.$$

Then, it holds:

$$D(\rho \| E_*^{\mathcal{M}}(\rho)) \leq \frac{\lambda_{\max}(\sigma)}{\lambda_{\min}(\sigma)} \left[D(\rho \| E_{1*}(\rho)) + D(\rho \| E_{2*}(\rho)) \right] + \lambda_{\max}(\sigma)\, d_{\mathcal{H}}\, d\,.$$

Main results

Approximate Tensorization via Pinching

As $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$, if $\mathcal{H} = \bigoplus_{i \in I_\mathcal{M}} \mathcal{H}_i \otimes \mathcal{K}_i$, then $\mathcal{M} = \bigoplus_{i \in I_\mathcal{M}} \mathcal{B}(\mathcal{H}_i) \otimes \mathbb{1}_{\mathcal{K}_i}$. Given a state ρ, consider $\mathcal{P}_{\rho_\mathcal{M}}$, the Pinching map with respect to $E_*^\mathcal{M}(\rho)$. We have:

$$D(\rho \| E_*^\mathcal{M}(\rho)) \leq \frac{1}{1-2c_1} \left[D(\rho \| E_{1*}(\rho)) + D(\rho \| E_{2*}(\rho)) \right] + \xi \left(E_{1*}(\rho), E_{2*}(\rho), E_*^\mathcal{M}(\rho) \right) ,$$

where

$$c_1 := \max_{i \in I_\mathcal{M}} \left\| E_1^{(i)} \circ E_2^{(i)} - (E^\mathcal{M})^{(i)} : \mathbb{L}_1(\tau_i) \to \mathbb{L}_\infty(\mathcal{N}) \right\| ,$$

and $\xi \left(E_{1*}(\rho), E_{2*}(\rho), E_*^\mathcal{M}(\rho) \right)$ strongly depends on $\mathcal{P}_{\rho_\mathcal{M}}$.

Important tools used in the proof:

- **Multivariate trace inequalities** (Sutter-Berta-Tomamichel '17)

$$\operatorname{tr}[\exp(H_1 + H_2 + H_3)] \leq \int_{-\infty}^{+\infty} dt\, \beta_0(t) \operatorname{tr}\left[e^{H_1} e^{\frac{1+it}{2} H_2} e^{H_3} e^{\frac{1-it}{2} H_2} \right] .$$

- **Chain rule for the relative entropy** (Ohya-Petz '04, Junge-Laracuente-Rouzé '20): If $\sigma = E_*(\sigma)$, then

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MLSI for Pinching onto different bases

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$\mathcal{N}_1$, $\mathcal{N}_2$ diagonal onto first and second basis, respectively. $\mathcal{M} = \mathbb{C}\mathbb{1}_\ell$.

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Then,

$$D(\rho \| \ell^{-1} \mathbb{1}) \leq \frac{1}{1 - 2\varepsilon} \left(D(\rho \| E_{1*}(\rho)) + D(\rho \| E_{2*}(\rho)) \right) ,$$

and subsequently

$$\mathcal{L}(X) := E_1(X) + E_2(X) - 2X .$$

has MLSI$(1 - 2\varepsilon)$.

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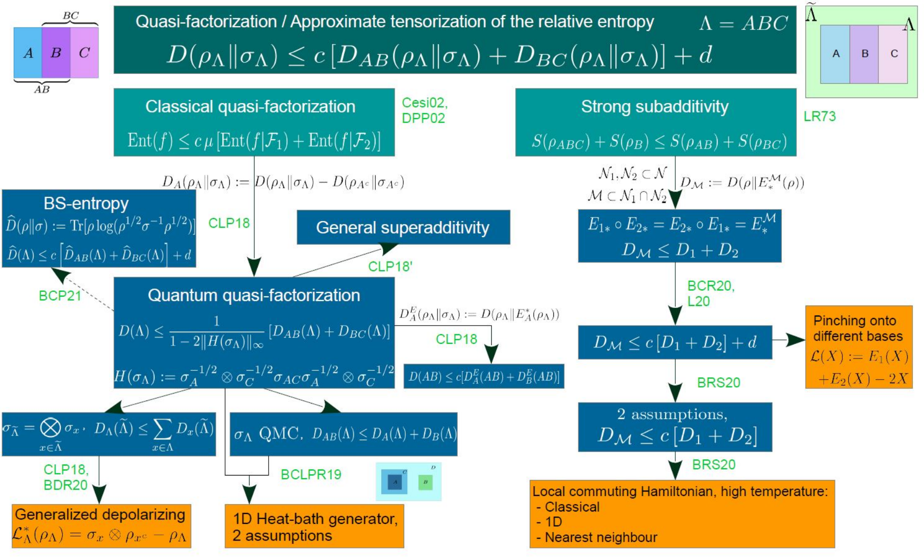
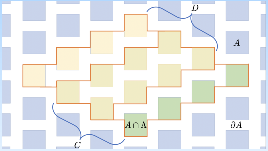


Approximate tensorization of the relative entropy



Approximate tensorization (C.-Rouzé-Stilck França '20)

Let $\mathcal{L}$ be a Gibbs sampler corresponding to a commuting potential. Assume further that the family $\mathcal{L}$ satisfies some conditions of clustering of correlations and on the fixed points of the generator. Then, for any $C, D \in \widetilde{\mathcal{S}}$ such that $C, D \subset \Lambda \subset\subset \mathbb{Z}^d$ with $2c\,|C \cup D| \exp\left(- \frac{\mathrm{d}(C\setminus D, D\setminus C)}{\xi}\right) < 1$, and all $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$,

$$D(\omega \| E_{C\cup D*}(\omega)) \leq \frac{1}{1 - 2c\,|C \cup D|\,\mathrm{e}^{-\frac{\mathrm{d}(C\setminus D, D\setminus C)}{\xi}}} \Big(D(\omega\|E_{C*}(\omega)) + D(\omega\|E_{D*}(\omega)) \Big),$$

with $\omega := E_{A\cap\Lambda*}(\rho)$.

Here, we show that a condition on the **fixed points** of the generator and a condition of **decay of correlations** imply

$$d = 0, \; c \sim 1 + \kappa\,\mathrm{e}^{-\,\mathrm{d}(C\setminus D, D\setminus C)} \; .$$

Tightened entropic uncertainty relations

Given two POVMs $\mathbf{X} := \{X_x\}_x$ and $\mathbf{Y} := \{Y_y\}_y$ on a quantum system A, and in the presence of side information M, for any bipartite state $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_M)$,

$$S(X|M)_{(\Phi_{\mathbf{X}} \otimes \mathrm{id}_M)(\rho)} + S(Y|M)_{(\Phi_{\mathbf{Y}} \otimes \mathrm{id}_M)(\rho)} \geq -\ln c' + S(A|M)_\rho \,,$$

with $c' = \max_{x,y}\{\mathrm{tr}(X_x Y_x)\}$, where $\Phi_{\mathbf{Z}}$ denotes the quantum-classical channel corresponding to the measurement $\mathbf{Z} \in \{\mathbf{X}, \mathbf{Y}\}$:

$$\Phi_{\mathbf{Z}}(\rho_A) := \sum_z \mathrm{tr}(\rho_A Z_z)\, |z\rangle\langle z|_Z \,.$$

Entropic uncertainty relation

(Frank-Lieb '13) Given a finite alphabet $\mathcal{Z} \in \{\mathcal{X}, \mathcal{Y}\}$, let $E_{\mathcal{Z}}$ denote the Pinching channels onto the orthonormal basis $\{|e_z^{(\mathcal{Z})}\rangle\}_{z\in\mathcal{Z}}$ corresponding to the measurement $\mathbf{Z}$. Assume further that $c_1 = d_A \max_{x,y} \left| |\langle e_x^{(\mathcal{X})} | e_y^{(\mathcal{Y})}\rangle|^2 - \frac{1}{d_A} \right| < 1$. Then the following *strenghtened entropic uncertainty relation* holds for any state $\rho \in \mathcal{D}(\mathcal{H}_A)$,

$$S(X)_{E_{\mathcal{X}}(\rho)} + S(Y)_{E_{\mathcal{Y}}(\rho)} \geq (1 + c_1)\, S(A)_\rho + (1 - c_1) \ln d_A \,.$$

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THANK YOU FOR YOUR ATTENTION!