On the entropic convergence of Gibbs samplers

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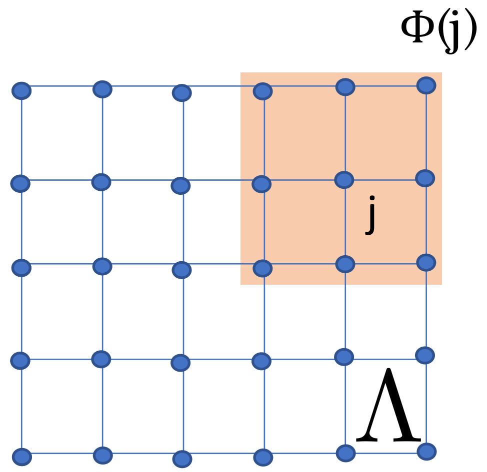
Quantum lattice systems in and out of equilibrium

?

Equilibrium properties

$$H_\Lambda := \sum_{j\in\Lambda} \Phi(j) \qquad \sigma_A = \frac{e^{-\beta\, H_A}}{\mathrm{Tr}\left(e^{-\beta H_A}\right)}$$

Ex: Ising $\quad \Phi(j) \in \{X, Z, ZZ\}$



Out-of-equilibrium properties

$$e^{t\mathcal{L}_\Lambda} \underset{t\to\infty}{\longrightarrow} \sigma_\Lambda$$

Davies generator (weak coupling to a bath)

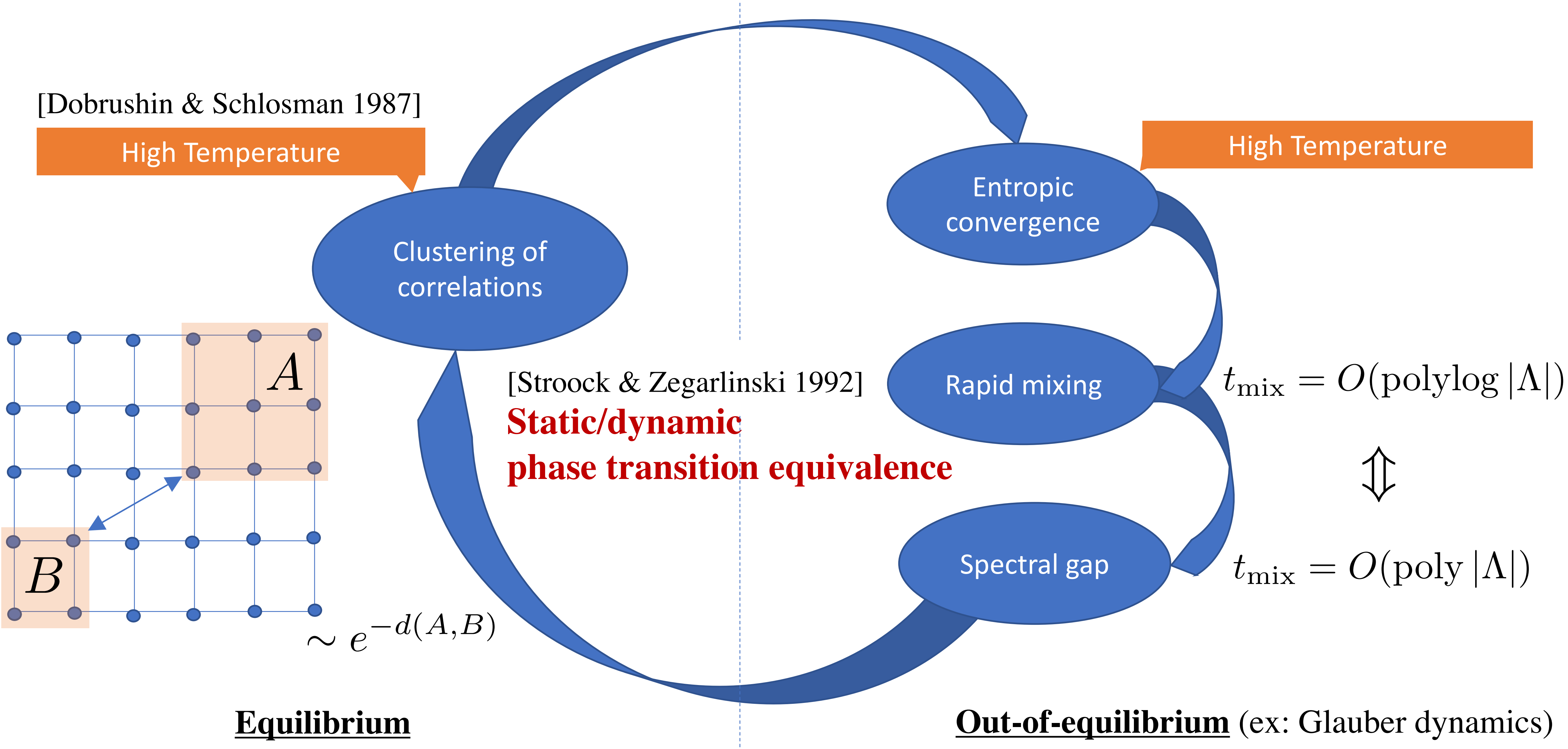
Heat bath (local loss+recovery w.r.t. σ_Λ)

Glauber dynamics

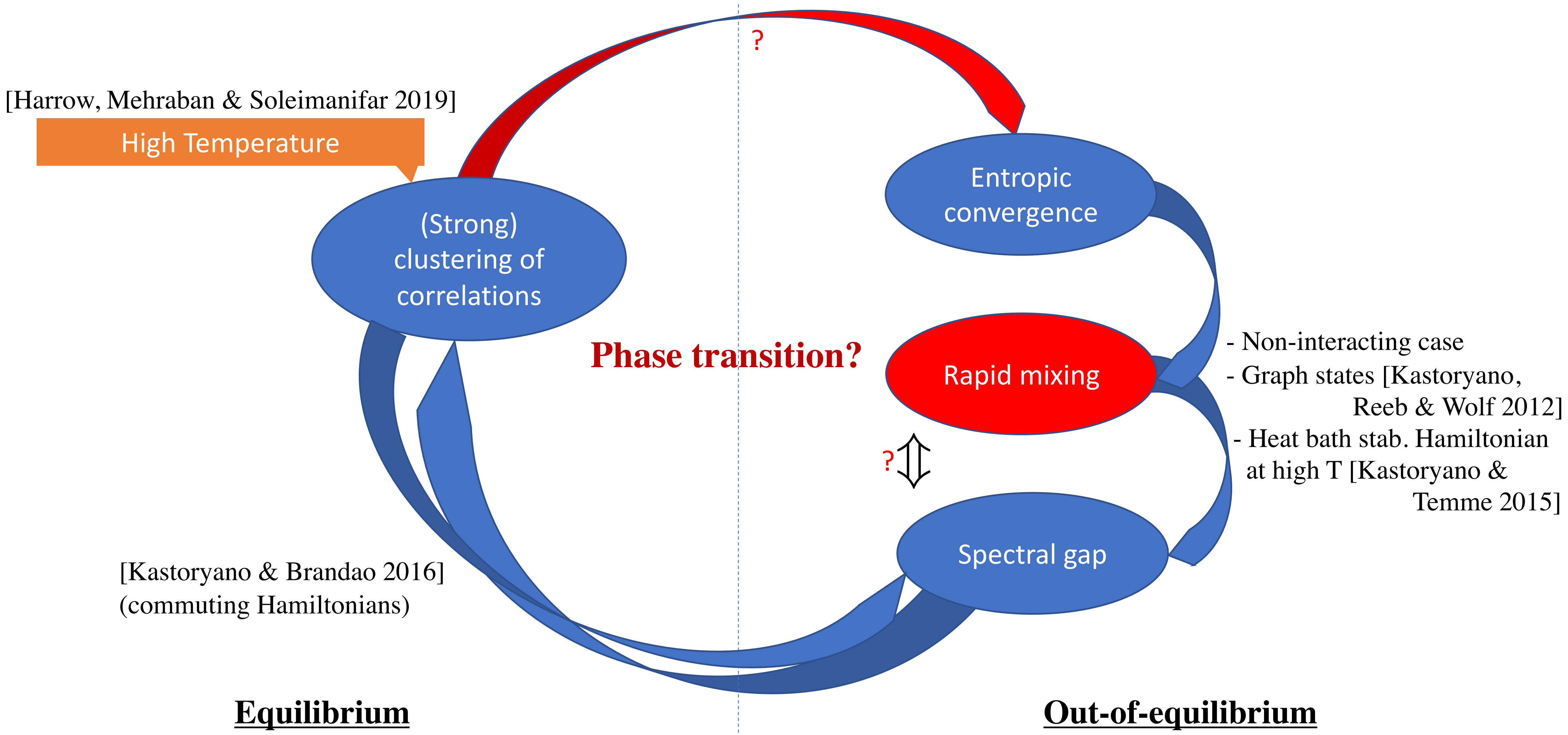
✓

Question: How do equilibrium properties of the system influence its thermalization?

Thermalization times in classical lattice systems



Thermalization times in quantum lattice systems



Entropic convergence: why care?

- Provide a neat proof of equivalence between static and dynamical phase transition

- Entropic convergence implies sharp finite sample size bounds on the limitations for the basic task of discrimination of two Gibbs states [Beigi, Datta, CR 2018]

- Limitations of optimization algorithms on noisy quantum devices were found for a variety of noise models using entropic convergence [Stilck França, Garcia-Patron 2020]

Main result

Given the Gibbs state σ_Λ of a local commuting Hamiltonian H_Λ, there exists a local quantum Markov semigroup converging to σ_Λ exponentially fast in relative entropy distance if

i H_Λ is classical for $\beta < \beta_c$;

ii H_Λ is a nearest neighbour Hamiltonian, for $\beta < \beta_c$;

iii Λ is 1D.

$$\forall \rho \in \mathcal{D}(\mathcal{H}_\Lambda),\; D(\rho_t \| \sigma_\Lambda) \leq e^{-\alpha\, t}\, D(\rho \| \sigma_\Lambda)$$

$\alpha \neq \text{fct}(|\Lambda|)$

$\Rightarrow$ First unconditional proof of decay in relative entropy for quantum lattice systems at high T.

$D(\rho \| \sigma) := \text{Tr}\left[\rho\,(\ln \rho - \ln \sigma)\right]$

Modified logarithmic Sobolev inequality

$(e^{t\mathcal{L}_\Lambda})_{t\geq 0}$: semigroup of quantum channels on $\mathcal{B}(\mathcal{H}_\Lambda)$, $C \subseteq \Lambda$, $\mathcal{L}_C = \sum_{j\in C} \mathcal{L}_j$, $\mathcal{L}_j(\sigma_\Lambda) = 0\, \forall j$

Assume detailed balance $\Rightarrow$ $e^{t\mathcal{L}_C}(\rho) \to E_C(\rho), t \to \infty$, $E_C : \mathcal{B}(\mathcal{H}_\Lambda) \to \mathrm{Ker}(\mathcal{L}_C)$

$E_\Lambda(\rho) = \sigma_\Lambda$

Entropic convergence: $\forall \rho \in \mathcal{D}(\mathcal{H}),\ D(e^{t\mathcal{L}_C}(\rho)\|E_C(\rho)) \leq e^{-\alpha\, t}\, D(\rho\|E_C(\rho))$

$$\Updownarrow \left.\frac{d}{dt}\right|_{t=0}$$

Modified logarithmic Sobolev inequality

$$\alpha D(\rho\|E_C(\rho)) \leq \mathrm{EP}_{\mathcal{L}_C}(\rho) := -\left.\frac{d}{dt}\right|_{t=0} D(e^{t\mathcal{L}_C}(\rho)\|E_C(\rho)) \equiv -\mathrm{Tr}[\mathcal{L}_C(\rho)(\ln(\rho) - \ln(\sigma_\Lambda))]$$

Relative entropy distance to $\mathrm{Ker}(\mathcal{L}_C)$

Entropy production linear in $\mathcal{L}_C$:

$$\mathrm{EP}_{\mathcal{L}_A+\mathcal{L}_B} = \mathrm{EP}_{\mathcal{L}_A} + \mathrm{EP}_{\mathcal{L}_B}$$

$\alpha(\mathcal{L}_\Lambda)$: MLSI constant

Non-interacting lattice systems

Tensorization property (classical systems):

$$\alpha(\mathcal{L}_A \otimes \mathrm{id}_B + \mathrm{id}_A \otimes \mathcal{L}_B) = \min\{\alpha(\mathcal{L}_A), \alpha(\mathcal{L}_B)\}$$

Does not hold for quantum systems [Brannan, Gao & Junge 2020]

Complete MLSI (CMLSI) constant [Gao, Junge & Laracuente 2019] $\alpha_c(\mathcal{L}) := \inf_{j \in \mathbb{N}} \alpha(\mathcal{L} \otimes \mathrm{id}_j)$

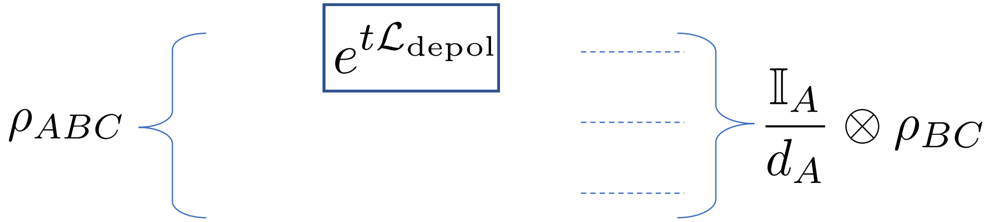
CMLSI satisfies tensorization

[Gao & CR 2021] [Gao, Junge & Li 2021] $\alpha_c(\mathcal{L}) > 0$ for finite matrix algebras.

Approaching interacting case via entropy inequalities

Tensorization (depolarizing semigroup) from **Strong subadditivity of entropy** [Lieb & Ruskai 1973]:

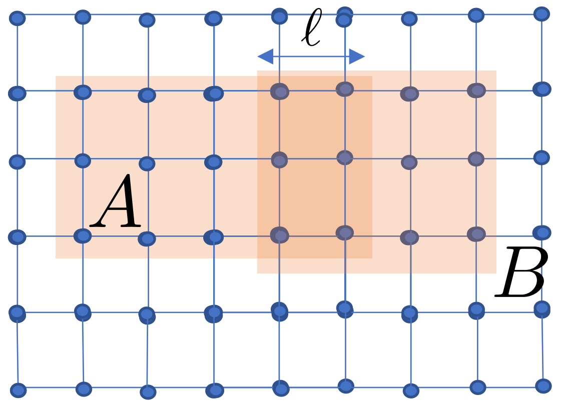
$$D(\rho\|\rho_C \otimes \frac{\mathbb{I}_{AB}}{d_{AB}}) \leq D(\rho\|\rho_{AC} \otimes \frac{\mathbb{I}_B}{d_B}) + D(\rho\|\rho_{BC} \otimes \frac{\mathbb{I}_A}{d_A})$$



Defining by $E_C := \lim_{t\to\infty} e^{t\mathcal{L}_C}$ the infinite time limit of the evolution restricted to region C,

$$E_A \circ E_B = E_B \circ E_A = E_{A\cup B}$$

$$\Leftrightarrow \quad D(\rho\|E_{A\cup B}(\rho)) \leq D(\rho\|E_A(\rho)) + D(\rho\|E_B(\rho))$$



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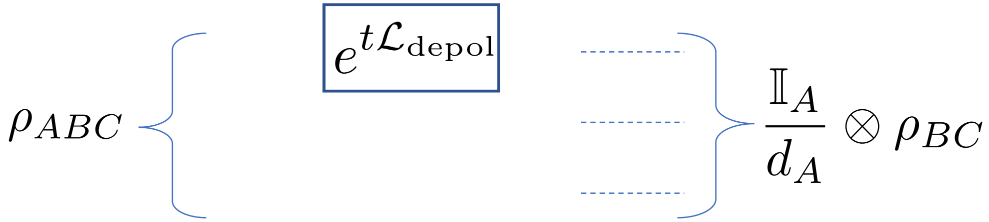

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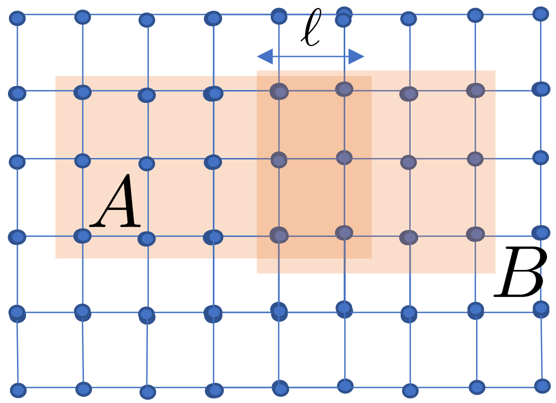

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$T < \infty$ [Cesi 2000]

$$D(\rho\|E_{A\cup B}(\rho)) \leq c\big[D(\rho\|E_A(\rho)) + D(\rho\|E_B(\rho))\big]$$

c related to correlations in the Gibbs state

$$c \sim 1 + \kappa\, e^{-\ell} \Rightarrow \alpha(\mathcal{L}_\Lambda) > 0$$

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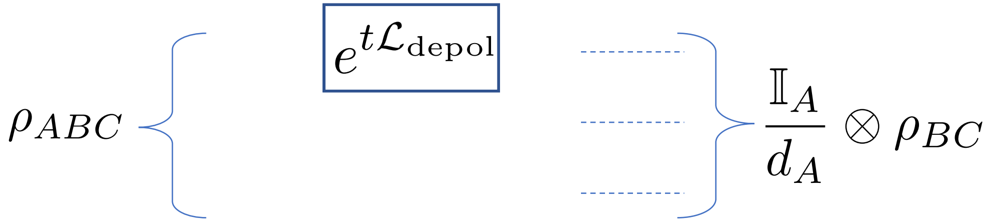

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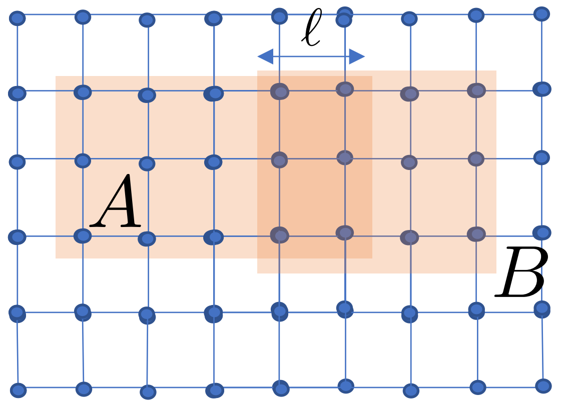

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$T < \infty$ [Cesi 2000] [Bardet Capel CR 2020] $D(\rho\|E_{A\cup B}(\rho)) \leq c\big[D(\rho\|E_A(\rho)) + D(\rho\|E_B(\rho))\big] + q$

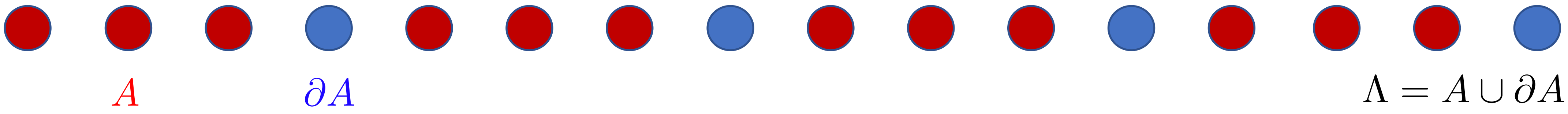
c related to correlations in the Gibbs state, q related to entanglement between $(A \cup B)^c$ and $A \cup B$, and coherences in boundary for ρ.

Strong approximate tensorization of relative entropy: $q = 0$ $c \sim 1 + \kappa\, e^{-\ell} \Rightarrow \alpha(\mathcal{L}_\Lambda) > 0$

Removing quantum properties: peeling out

Intuition: $\tau_{\mathrm{decoherence}}, \tau_{\mathrm{entanglement}} << \tau_{\mathrm{thermalization}}$

Example: σ_Λ classical state, $\mathcal{L}_\Lambda$ classical Glauber dynamics, 1D nearest neighbour interactions:



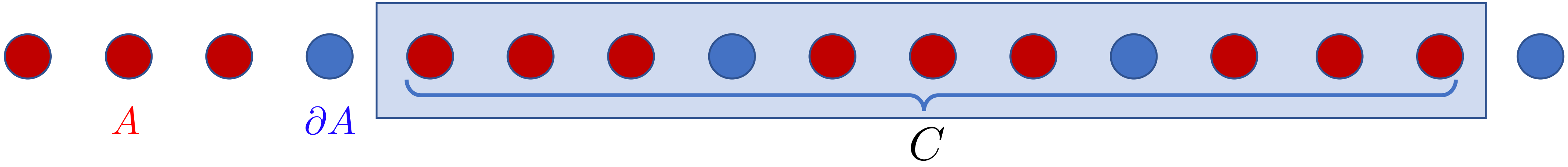
$$E_A(\rho) = \sum_{\omega_{\partial A}} \mathrm{Tr}[\langle \omega_{\partial A} | \rho | \omega_{\partial A} \rangle] \, |\omega_{\partial A}\rangle\langle \omega_{\partial A}| \otimes \sigma_A^{\omega_{\partial A}}$$

Classical state

$$D(\rho \| \sigma_\Lambda) = D(\rho \| E_A(\rho)) + D(E_A(\rho) \| \sigma_\Lambda) \qquad \text{(Chain rule)}$$

Pinched MLSI

Cf: 1D chain, nearest neighbour interactions



Definition (Pinched MLSI):

$$\gamma_C \, D(E_A(\rho) \| E_C \circ E_A(\rho)) \leq \mathrm{EP}_{\mathcal{L}_C}(\rho)$$

$$\begin{aligned} D(\rho \| E_\Lambda(\rho)) &= D(\rho \| E_A(\rho)) + D(E_A(\rho) \| E_\Lambda(\rho)) && \text{(Chain rule)} \\ &\leq \alpha_c(\mathcal{L}_{A^*})^{-1} \, \mathrm{EP}_{\mathcal{L}_A}(\rho) + {\gamma_\Lambda}^{-1} \, \mathrm{EP}_{\mathcal{L}_\Lambda}(\rho) && \text{(CMLSI + Pinched MLSI)} \\ &\leq \left(\alpha_c(\mathcal{L}_{A^*})^{-1} + {\gamma_\Lambda}^{-1}\right) \mathrm{EP}_{\mathcal{L}_\Lambda}(\rho) && \text{(linearity of EP)} \end{aligned}$$

Question: Is γ_Λ independent of the lattice size?

Controlling the Pinched MLSI constant via decay of correlations

Pinched MLSI: $$\gamma_C \, D(\rho_A \| E_C(\rho_A)) \leq \mathrm{EP}_{\mathcal{L}_C}(\rho) \, , \qquad \rho_A := E_A(\rho)$$

[Dobrushin & Schlosman 1987] Gibbs measures satisfy a **decay of correlations** in 1D or below β_c

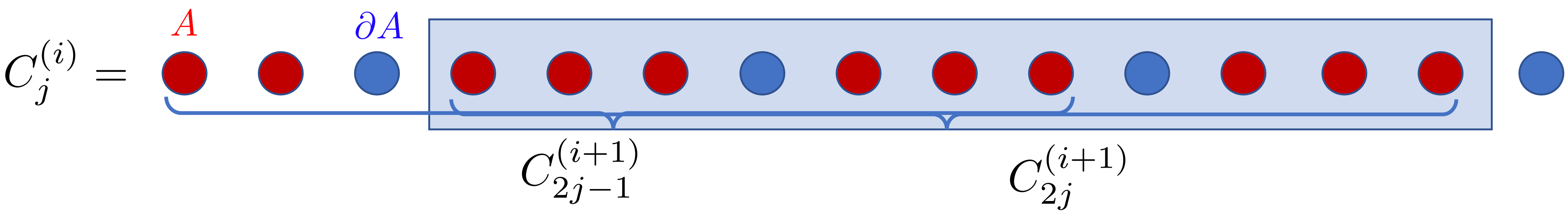
$$\| E_C^{\omega_{\partial C \cup D}} \circ E_D^{\omega_{\partial C \cup D}} - E_{C \cup D}^{\omega_{\partial C \cup D}} : L^1(\sigma_{C \cup D}^{\omega_{\partial C \cup D}}) \to L^\infty \| \leq c \, |C \cup D| \, e^{-\mathrm{dist}(C^c, D^c)/\xi}$$

[Cesi 2001] For all classical state p, for $\mathrm{dist}(C^c, D^c) \equiv \ell$ large enough

$$D(p \| E_{C \cup D}(p)) \leq (1 + e^{-\kappa \ell}) \left[D(p \| E_C(p)) + D(p \| E_D(p)) \right] (\star)$$

Since ρ_A is classical, we can iterate $(\star)$:

$$D(\rho_A \| E_\Lambda(\rho_A)) \leq (1 + K) \sum_{j=1}^{2^N} D(\rho_A \| E_{C_j^{(N)}}(\rho_A))$$



Bounding the Pinched MLSI constant

Pinched MLSI: $\gamma_C \, D(\rho_A \| E_C(\rho_A)) \leq \mathrm{EP}_{\mathcal{L}_C}(\rho)\,, \qquad \rho_A := E_A(\rho)$

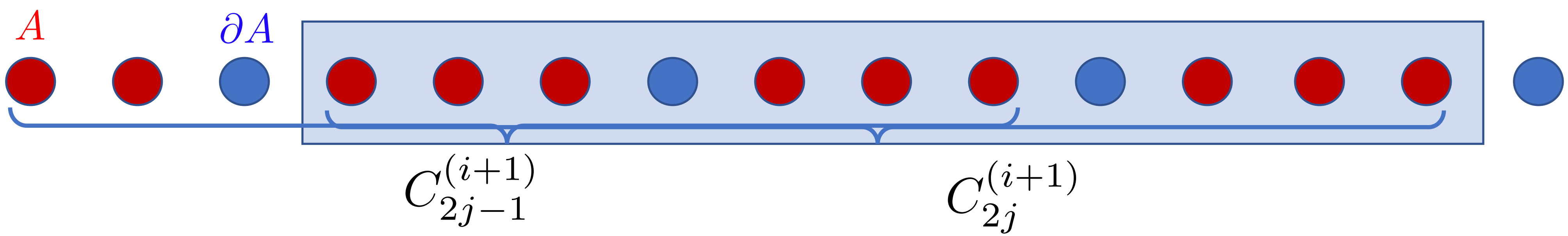
We proved: $D(\rho_A \| E_\Lambda(\rho_A)) \leq (1+K) \sum_{j=1}^{2^N} D(\rho_A \| E_{C_j^{(N)}}(\rho_A))$

$$
\begin{aligned}
&\leq \gamma_C^{-1} \sum_{j=1}^{2^N} \mathrm{EP}_{\mathcal{L}_{C_j^{(N)}}}(\rho) \\
&\leq 2\,\gamma_C^{-1} \; \mathrm{EP}_{\mathcal{L}_\Lambda}(\rho) \\
&\leq 2\,\alpha_c(\mathcal{L}_C)^{-1} \; \mathrm{EP}_{\mathcal{L}_\Lambda}(\rho)
\end{aligned}
$$

Proof of $\gamma_C \geq \alpha_c(\mathcal{L}_C)$**:** Since $[E_C, E_A] = 0\,,$

$$
\begin{aligned}
D(\rho_A \| E_C(\rho_A)) &= D(E_A(\rho) \| E_A \circ E_C(\rho)) \\
&\leq D(\rho \| E_C(\rho)) \\
&\leq \alpha_c(\mathcal{L}_C)^{-1} \; \mathrm{EP}_{\mathcal{L}_C}(\rho)
\end{aligned}
$$

$C_j^{(i)} =$



Conclusion

Our results:

- First unconditional proof of MLSI on quantum lattice systems for classical, nearest neighbour commuting and 1D commuting Hamiltonians

Tools:

- Generalizations of SSA: approximate tensorization of the relative entropy for certain classes of states

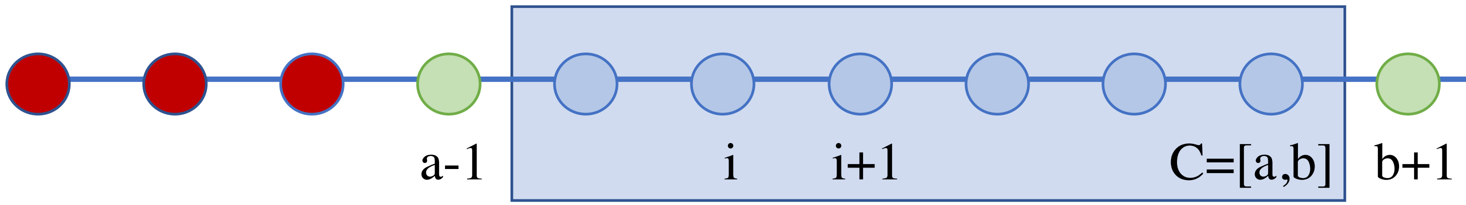
- Introduction of Pinched MLSI to restrict the analysis to that exact class of states

Open problem:

- Extending the result to k-local commuting Hamiltonians? [Gao & CR 2021]

Thank you for your attention!

Nearest neighbour, commuting Hamiltonians



[Bravyi & Vyalyi 2005]

$$e^{-\beta H_\Lambda} = \prod_{j\in\Lambda} e^{-\beta h_{i,i+1}}$$, where $$e^{-\beta h_{i,i+1}} = \sum_\ell X^\ell_{i,R} \otimes X^\ell_{i+1,L}$$ (Schmidt decomposition)

$$\mathcal{A}_{[a,b]} := \mathcal{B}(\mathcal{H}_{\leq a-2}) \otimes \mathcal{A}_{a-1,L} \otimes I_{[a,b]} \otimes \mathcal{A}_{b+1,R} \otimes \mathcal{B}(\mathcal{H}_{\geq b+2}), \quad E^*_{[a,b]} : \mathcal{B}(\mathcal{H}_\Lambda) \to \mathcal{A}_{[a,b]}$$

- Refinement of [Harrow, Mehraban, Soleimanifar 2019] **conditional decay of correlations.**
- Refinement of [Bardet, Capel, CR 2020] **approximate tensorization of RE:**

$$\forall \rho = E_A(\rho),\ D(\rho \| E_{C\cup D}(\rho)) \leq (1 + \mathcal{O}(e^{-d(C^c,D^c)})) \left(D(\rho\|E_C(\rho)) + D(\rho\|E_D(\rho))\right)$$

- The argument for Glauber dynamics extends for the generator

$$\mathcal{L}_\Lambda = \sum_{j\in\Lambda} E_j - \mathrm{id}$$