On the data processing inequality for the relative entropy between two quantum states

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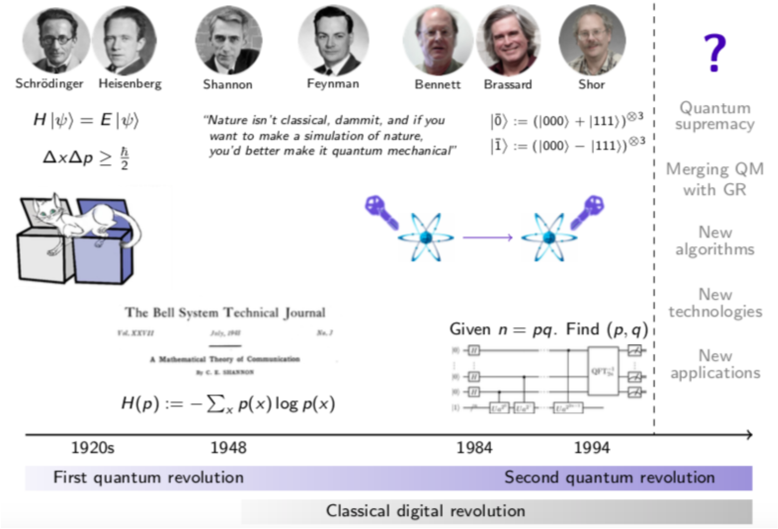
Joint work with Andreas Bluhm (U. Copenhagen)

Based on arXiv: 1904.10768

V Congreso de Jóvenes Investigadores de la RSME, 28th January 2020

Origins of Quantum Information Theory



Acknowledgement: David Sutter (IBM Research)

Classical Physics versus Quantum Physics

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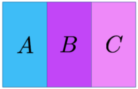

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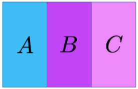

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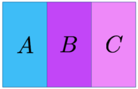

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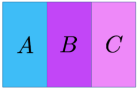

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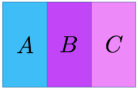

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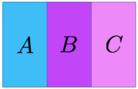

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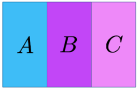

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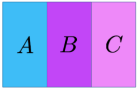

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Moreover, this is **robust**:

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for Q_{XYZ} a Markov chain.

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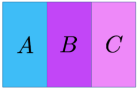

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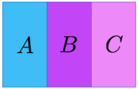

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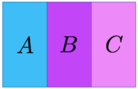

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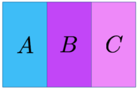

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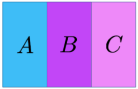

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Quantum channel: $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ CPTP map.

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Petz recovery map: $\mathcal{R}_{\mathcal{T}}^{\rho}(\cdot) := \rho^{1/2}\mathcal{T}^* \left(\mathcal{T}(\rho)^{-1/2}(\cdot)\mathcal{T}(\rho)^{-1/2} \right) \rho^{1/2}$.

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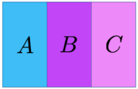

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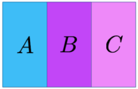

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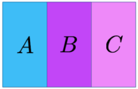

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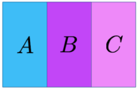

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Operational meaning of $D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho))$

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Extensions and improvements of the previous result:
$D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \geq (1), (2), (3)$, where:

$$(1) := -\int \beta_0(t) \log F\left(\sigma, \mathcal{R}_{\mathcal{T}}^{\rho,[t]} \circ \mathcal{T}(\sigma)\right) \mathrm{d}t \text{ (Junge et al. '15)},$$

with

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Motivation: Strengthened bounds for DPI of RE

Extensions and improvements of the previous result:
$D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \geq (1), (2), (3)$, where:

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Can we find a lower bound for the DPI in terms of $D(\sigma || \mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma))$*?*

Answer: It is not possible **(Brandao et al. '15, Fawzi**[2] **'17)**.

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(Sutter-Renner '18) $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, $\sigma_{ABC} > 0$ and $\rho_{ABC} = \mathbb{1}_A / d_A \otimes \sigma_{BC}$, $\mathcal{T}(\cdot) = \text{tr}_C[\cdot]$.

$$D(\sigma_{ABC} || \mathcal{R}^{\sigma_{BC}}_{\text{tr}_C} \circ \text{tr}_C[\sigma_{ABC}]) + \Lambda_{\max}(\sigma_{AB} || \mathcal{R}_{B \to B}) \geq I(A : C|B)_{\sigma},$$

where

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Can we find a lower bound for the DPI in terms of $\mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma)$?

(Carlen-Vershynina '17) $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ conditional expectation, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$:

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(Carlen-Vershynina '18) Extension to standard f-divergences.

Some definitions

Conditional expectation

Let $\mathcal{M}$ matrix algebra with matrix subalgebra $\mathcal{N}$. There exists a unique linear mapping $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ such that

1. $\mathcal{E}$ is a positive map,
2. $\mathcal{E}(B) = B$ for all $B \in \mathcal{N}$,
3. $\mathcal{E}(AB) = \mathcal{E}(A)B$ for all $A \in \mathcal{M}$ and all $B \in \mathcal{N}$,
4. $\mathcal{E}$ is trace preserving.

A map fulfilling (1)-(3) is called a *conditional expectation*.

Belavkin-Staszewski relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **BS-entropy** is defined as:

$$\hat{S}_{\mathrm{BS}}(\sigma||\rho) := \operatorname{tr}\left[\sigma \log\left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right].$$

Relation between relative entropies

The following holds for every $\sigma > 0, \rho > 0$:

$$\hat{S}_{\mathrm{BS}}(\sigma||\rho) \geq D(\sigma||\rho).$$

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Some definitions

Operator convex

Let $\mathcal{I} \subseteq \mathbb{R}$ interval and $f : \mathcal{I} \to \mathbb{R}$. If

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$

for all Hermitian A, $B \in \mathcal{B}(\mathcal{H})$ with spectrum contained in $\mathcal{I}$, all $\lambda \in [0,1]$, and for all finite-dimensional Hilbert spaces $\mathcal{H}$, then f is *operator convex*.

Standard f-divergences

(Hiai-Mosonyi '17)

Standard f-divergences

Let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$S_f(\sigma \| \rho) = \operatorname{tr}\left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) \rho^{1/2}\right]$$

is the *standard f-divergence.*

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Example: Let $f(x) = x \log x$. Then,

$$S_f(\sigma\|\rho) = \operatorname{tr}[\sigma(\log \sigma - \log \rho)]$$

defines the relative entropy $D(\sigma\|\rho)$.

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Conditions for equality

Let $\sigma > 0$, $\rho > 0$ be on $\mathcal{M}$ and let $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a 2PTP linear map. Then, the following are equivalent:

1. There exists a TP map $\hat{\mathcal{T}} : \mathcal{B} \to \mathcal{M}$ such that $\hat{\mathcal{T}}(\mathcal{T}(\rho)) = \rho$ and $\hat{\mathcal{T}}(\mathcal{T}(\sigma)) = \sigma$.
2. $S_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) = S_f(\sigma \| \rho)$ for all operator convex f on $[0, \infty)$.
3. $\mathcal{R}^{\rho}_{\mathcal{T}}(\mathcal{T}(\sigma)) = \sigma$.

Maximal f-divergences

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Let $f : (0,\infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

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Relation between f-divergences

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For every two states $\sigma > 0$, $\rho > 0$ on $\mathcal{M}$ and every operator convex function $f : (0, \infty) \to \mathbb{R}$,

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Remark: Difference

For maximal f-divergences, there is no equivalent condition for equality in DPI which provides a explicit expression of recovery for σ.

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Questions

BS recovery condition

Can we prove an equivalent condition for equality in DPI for the BS entropy (or for maximal f-divergences) which provides a explicit expression of recovery for σ?

Strengthened DPI for BS entropy

Following Carlen-Vershynina, can we provide a lower bound for the DPI for the BS entropy (or for maximal f-divergences) in terms of a (hypothetical) BS recovery condition?

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Equivalent conditions for equality on DPI

$\Gamma := \sigma^{-1/2}\rho\sigma^{-1/2}$ and $\Gamma_{\mathcal{T}} := \sigma_{\mathcal{T}}^{-1/2}\rho_{\mathcal{T}}\sigma_{\mathcal{T}}^{-1/2}$
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Equivalent conditions for equality on DPI (Bluhm-C. '19)

Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on $\mathcal{M}$. Then, the following are equivalent:

1. $\hat{S}_{\mathrm{BS}}(\sigma\|\rho) = \hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}})$.
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Consequences

Note: Although they can be seen as a consequence of the previous result, the following facts were previously known.

Corollary

$$\begin{aligned}\hat{S}_{\mathrm{BS}}(\sigma\|\rho) = \hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}) &\Leftrightarrow \rho = \mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho) \\ &\Leftrightarrow \sigma = \mathcal{B}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma) \\ &\Leftrightarrow \hat{S}_{\mathrm{BS}}(\rho\|\sigma) = \hat{S}_{\mathrm{BS}}(\rho_{\mathcal{T}}\|\sigma_{\mathcal{T}}).\end{aligned}$$

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$$D(\sigma\|\rho) = D(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}) \implies \hat{S}_{BS}(\sigma\|\rho) = \hat{S}_{BS}(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}).$$

Equivalently,

$$\sigma = \mathcal{R}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma) \implies \sigma = \mathcal{B}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma).$$

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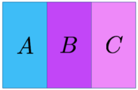
Results for the BS-entropy, Bluhm-C. '19

Relative entropy	**BS-entropy**
$\operatorname{tr}[\sigma(\log\sigma - \log\rho)]$	$\operatorname{tr}\left[\sigma \log\left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right]$
$\rho = \rho^{1/2}\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1/2}\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1/2}\right)\rho^{1/2}$	$\sigma = \rho\,\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1}\mathcal{T}(\sigma)\right)$
$\left(\frac{\pi}{8}\right)^4 \lVert L_\rho R_{\sigma^{-1}}\rVert_\infty^{-2}\left\lVert \mathcal{R}_{\mathcal{E}}^{\sigma}(\rho_{\mathcal{N}}) - \rho\right\rVert_1^4$	$\left(\frac{\pi}{8}\right)^4 \lVert\Gamma\rVert_\infty^{-4}\left\lVert\sigma^{-1}\right\rVert_\infty^{-2}\left\lVert\rho - \mathcal{B}_{\mathcal{T}}^{\sigma}\circ\mathcal{T}(\rho)\right\rVert_2^4$
Extension to standard f-divergences	Extension to maximal f-divergences

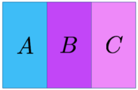
Future work



Particular case: $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Quantum channel: $\mathcal{T} = \mathrm{tr}_C$.

Future work



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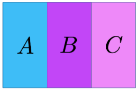

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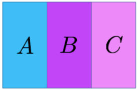

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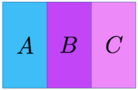

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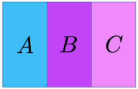
Future work

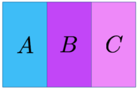


(Bluhm-C. '20)

$$\boxed{\sigma_{ABC} = \sigma_{BC}^{1/2}\,\sigma_B^{-1/2}\,\sigma_{AB}\,\sigma_B^{-1/2}\,\sigma_{BC}^{1/2}} \quad \begin{matrix}\Rightarrow \\ \not\Leftarrow\end{matrix} \quad \boxed{\sigma_{ABC} = \sigma_{BC}\,\sigma_B^{-1}\,\sigma_{AB}}$$

Define a **BS quantum state** as a state $\sigma_{ABC} \in \mathcal{S}_{ABC}$ such that $\sigma_{ABC} = \sigma_{BC}\,\sigma_B^{-1}\,\sigma_{AB}$.

Future work



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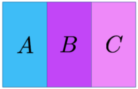
Question

Is the set of BS quantum states robust?

Future work



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References

A. Bluhm, A. Capel.
A strengthened data processing inequality for the Belavkin-Staszewski relative entropy
Reviews in Mathematical Physics, to appear (2019).

E. Carlen, A. Vershynina.
Recovery map stability for the Data Processing Inequality
Journal of Physics A: Mathematical and Theoretical, 53 (3), 035204, 2020.

O. Fawzi, R. Renner.
Quantum conditional mutual information and approximate Markov chains
Communications in Mathematical Physics, 340 (2) (2015), 575-61.

F. Hiai, M. Mosonyi.
Different quantum f-divergences and the reversibility of quantum operations
Reviews in Mathematical Physics, 29 (7) (2017).


thank you
danke
рахмат
謝謝
ngiyabonga
شكراً جزيلاً
Баярлалаа
спасибо
faafetai lava
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dankie
dhanyavad
nanni
nandri
bedankt
enkosi
bayarlalaa
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shukriya
mercê
мерси
তোমাকে ধন্যবাদ
감사합니다
xièxie
ευχαριστώ
merci