Quasi-factorization of the quantum relative entropy of two quantum states

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Objectives

1. We present a result of quasi-factorization of the quantum relative entropy of two quantum states.
2. This result will be further used to prove a sufficient condition for the property of rapid mixing of a quantum many body system.

Introduction

- In quantum information theory, a challenging problem is understanding interacting quantum many body systems and engineering and exploiting such quantum systems.
- We identify the particles of a quantum many body system with a lattice $\Lambda \subseteq \mathbb{Z}^d$. Each particle x has an associated Hilbert space $\mathcal{H}_x$ and the global Hilbert space is given by $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$. An element $\rho \in \mathcal{S}_\Lambda$, the subset of $\mathcal{B}(\mathcal{H}_\Lambda) \equiv \mathcal{B}_\Lambda$ whose elements are positive and with unital trace, is called a **density matrix**.
- For two density matrices ρ and σ, the **quantum relative entropy** of ρ with respect to σ is defined by
$$D(\rho||\sigma) = \operatorname{tr}[\rho(\log\rho - \log\sigma)]$$
and represents a measure of our ability to distinguish two quantum states.

Conditional relative entropy

Let $\mathcal{A}$ and $\mathcal{B}$ be two matrix algebras, and σ a full rank state on $\mathcal{A}\otimes\mathcal{B}$. A map $\mathbb{E} : \mathcal{A}\otimes\mathcal{B} \to \mathcal{A}$ is called a **conditional expectation for** σ if:

1. *(Complete positivity)* $\mathbb{E}$ is completely positive and unital.
2. *(Consistency)* For every $f \in \mathcal{A}\otimes\mathcal{B}$, $\operatorname{tr}[\sigma\mathbb{E}(f)] = \operatorname{tr}[\sigma f]$. In other words $\mathbb{E}^*(\sigma) = \sigma$, where the dual is taken with respect to the Hilber-Schimdt scalar product.
3. *(Reversibility)* For every $f, g \in \mathcal{A}\otimes\mathcal{B}$, $\langle \mathbb{E}(f), g\rangle_\sigma = \langle f, \mathbb{E}(g)\rangle_\sigma$. In other words, E is self-adjoint in $L_2(\sigma)$.
4. *(Monotonicity)* For every $f \in \mathcal{A}\otimes\mathcal{B}$ and $n \in \mathbb{N}$, $\langle \mathbb{E}^n(f), f\rangle_\sigma \geq \langle \mathbb{E}^{n+1}(f), f\rangle_\sigma$.

We define the **minimal conditional expectation** of σ on A by
$$\mathbb{E}_A^\sigma(\rho) := \operatorname{tr}_A[\eta_A^\sigma \rho \eta_A^{\sigma\dagger}],$$
where $\eta_A^\sigma := (\operatorname{tr}_A[\sigma])^{-1/2}\sigma^{1/2}$ and the partial trace $\operatorname{tr}_A : \mathcal{B}(\mathcal{H}_A\otimes\mathcal{H}_B) \to \mathcal{B}(\mathcal{H}_B)$ is the unique linear map such that $\operatorname{tr}_A(a\otimes b) = b\operatorname{tr}(a)$ for all $a \in \mathcal{B}(\mathcal{H}_A)$ and $b \in \mathcal{B}(\mathcal{H}_B)$.

Notice that $\mathbb{E}_A^*$ is given by
$$\mathbb{E}_A^*(\rho_B) := \sigma^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma^{1/2},$$
We define the following **conditional relative entropy**:
$$\boxed{D_A(\rho||\sigma) = D(\rho||\mathbb{E}_A^*(\rho_B))}.$$

The result

Let $A, B \subseteq \Lambda$ so that $A\cup B = \Lambda$ but they are not necessarily disjoint. Let ρ and σ be density matrices acting on $\mathcal{H}_\Lambda$. Then the following holds:
$$\boxed{(1 - 2\|h\|_\infty)D(\rho||\sigma) \leq D_A(\rho||\sigma) + D_B(\rho||\sigma)},$$
where $h = \sigma_{A^c}^{-1/2}\otimes\sigma_{B^c}^{-1/2}\sigma_{A^c\cup B^c}\sigma_{A^c}^{-1/2}\otimes\sigma_{B^c}^{-1/2} - \mathbb{1}$.

Note that $h = 0$ if $A\cap B = \emptyset$ and σ is a product.

Proof

We divide the proof in several steps:

- **1.** $\boxed{D(\rho||\sigma) \leq D_A(\rho||\sigma) + D_B(\rho||\sigma) + \log\operatorname{tr} M}$,
  where $\log M = -\log\sigma + \log\mathbb{E}_A^*(\rho_{A^c}) + \log\mathbb{E}_B^*(\rho_{B^c})$.
  In the proof of this step, we use the following lemma.
  - **Lemma.** For arbitrary positive matrices X and Y, it holds that
  $$D(X||Y) \geq -\log\frac{\operatorname{tr} Y}{\operatorname{tr} X}.$$
- **2.** $\boxed{\log\operatorname{tr} M \leq \log\operatorname{tr}\left[\mathbb{E}_A^*(\rho_{A^c})\mathcal{T}_\sigma(\mathbb{E}_B^*(\rho_{B^c}))\right]}$,
  where $\mathcal{T}_B(A) = \int_0^\infty \mathrm{d}t\,(B+t)^{-1}A(B+t)^{-1}$.
  We use the following theorem of Lieb, which constitutes an extension of the Golden-Thompson inequality.
  - **Theorem (Lieb).** For every operator A, B and C,
  $$\operatorname{tr}\exp(-A + B + C) \leq \operatorname{tr} e^C\mathcal{T}_{e^A}(e^B).$$
- **3.** $\boxed{\log\operatorname{tr} M \leq \operatorname{tr}[h\,\rho_{A^c}\otimes\rho_{B^c}]}$,
  where $h = \sigma_{A^c}^{-1/2}\otimes\sigma_{B^c}^{-1/2}\sigma_{A^c\cup B^c}\sigma_{A^c}^{-1/2}\otimes\sigma_{B^c}^{-1/2} - \mathbb{1}$.
  Among other things, we use the following lemma and the fact that $\log x \leq x - 1$.
  - **Lemma.** Let ρ and σ be two states. The following inequality holds:
  $$T_\sigma(\rho) \leq \frac{1}{2}\{\sigma^{-1}, \rho\} \;,$$
  where $\{A, B\}$ is the anticommutator of A and B, i.e., $\{A, B\} = AB + BA$.
- **4.** $\boxed{\log\operatorname{tr} M \leq 2\|h\|_\infty D(\rho||\sigma)}$,
  In the proof of this last step, we use Pinsker's inequality and the Data-Processing inequality.
  - **Pinsker's inequality.** For ρ and σ density matrices, it holds that
  $$\|\rho - \sigma\|_1^2 \leq 2D(\rho||\sigma).$$
  - **Data-Processing inequality.** For every quantum channel T, the following inequality holds:
  $$D(T(\rho)||T(\sigma)) \leq D(\rho||\sigma)\;.$$
  We also use the following lemma.
  - **Lemma.** The definition of h implies that for every operator O_{A^c} on A^c and O_{B^c} on B^c,
  $$\operatorname{tr}[h\,\sigma_{A^c}\otimes O_{B^c}] = \operatorname{tr}[h\,O_{A^c}\otimes\sigma_{B^c}] = 0.$$

Current work

- We want to study properties of quantum dissipative evolutions of spin systems on lattices which satisfy **rapid mixing** (fast convergence to the fixed point).
- For finding systems with rapid mixing, we want to characterize it by a property of decay of correlations on the fixed point of the evolution.
- Let $\mathcal{L}$ be the generator of the evolution of the system. We define the **log-Sobolev constant** of $\mathcal{L}$ by
$$S_\Lambda(\mathcal{L}) := \inf_{\rho\in\mathcal{S}_\Lambda}\frac{-\operatorname{tr}[\mathcal{L}^*(\rho)(\log\rho - \log\sigma)]}{2D(\rho||\sigma)}$$
We know that if this constant is positive, the system satisfies rapid mixing.
- Therefore, if σ is the fixed point of the evolution on Λ and if we assume the following mixing condition: For every $A, B \subseteq \Lambda$ such that $A\cap B = \emptyset$, there exist constants C_1, $C_2 > 0$ such that:
$$\sup_{\sigma\in\mathcal{S}_\Lambda}\left\|\sigma_A^{-1/2}\otimes\sigma_B^{-1/2}\sigma_{A\cup B}\sigma_A^{-1/2}\otimes\sigma_B^{-1/2} - \mathbb{1}\right\|_\infty \leq C_1 e^{-C_2\operatorname{dist}(A,B)},$$
we want to prove that the system has a positive log-Sobolev constant.
- For proving the existence of a positive global log-Sobolev constant, we define a **conditional log-Sobolev constant** on $A \subseteq \Lambda$, which we denote by $S_\Lambda(\mathcal{L}_A)$, and express the value of $S_\Lambda(\mathcal{L})$ in terms of $S_\Lambda(\mathcal{L}_A)$ and $S_\Lambda(\mathcal{L}_B)$, where $A\cup B = \Lambda$. For that, we use the quasi-factorization lemma we have just presented.
- Using a recursive argument, we can lower bound $S_\Lambda(\mathcal{L})$ by a constant times $S_\Lambda(\mathcal{L}_A)$, with A finite, what provides the possitiveness of the global log-Sobolev constant.

References

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