Weak Quasi-Factorization for the Belavkin-Staszewski Relative Entropy

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Belavkin-Staszewski relative entropy

(Umegaki) relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space and ρ_{AB}, σ_{AB} two positive states on it. Their *Umegaki relative entropy* is given by:

$$D(\rho_{AB}\|\sigma_{AB}) := \mathrm{tr}[\rho_{AB}(\log \rho_{AB} - \log \sigma_{AB})] \,.$$

Belavkin-Staszewski relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space and ρ_{AB}, σ_{AB} two positive states on it. Their *Belavkin-Staszewski relative entropy* (BS-entropy for short) is given by:

$$\widehat{D}(\rho_{AB}\|\sigma_{AB}) := \mathrm{tr}\left[\rho_{AB} \log\left(\rho_{AB}^{1/2}\sigma_{AB}^{-1}\rho_{AB}^{1/2}\right)\right] .$$

Relation

The following holds for any positive states ρ_{AB}, σ_{AB}:

$$D(\rho_{AB}\|\sigma_{AB}) \leq \widehat{D}(\rho_{AB}\|\sigma_{AB}) \,,$$

and the inequality is strict if, and only if, $[\rho_{AB}, \sigma_{AB}] \neq 0$.

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Motivation: Quasi-factorization relative entropy

Given a positive state σ_{AB}, denote:

$$H(\sigma_{AB}) := \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \, .$$

QF relative entropy (C.-Lucia-Pérez García '18)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following inequality holds whenever $\|H(\sigma_{AB})\|_\infty < 1/2$:

$$D(\rho_{AB}||\sigma_{AB}) \leq \frac{1}{1 - 2\|H(\sigma_{AB})\|_\infty} \left[D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}) \right] ,$$

where

$$D_A(\rho_{AB}||\sigma_{AB}) := D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B) \, .$$

Applications:

- It is equivalent to a generalization of **superadditivity**.
- Key tool to prove positivity of **modified logarithmic Sobolev inequalities** for quantum many-body systems.

Weak quasi-factorization for the BS-entropy

Weak quasi-factorization BS-entropy

It is an inequality of the form:

$$\widehat{D}(\rho_{AB}||\sigma_{AB}) \leq M\left[\widehat{D}_A(\rho_{AB}||\sigma_{AB}) + \widehat{D}_B(\rho_{AB}||\sigma_{AB})\right] + L\,,$$

where $$\widehat{D}_A(\rho_{AB}||\sigma_{AB}) := \widehat{D}(\rho_{AB}||\sigma_{AB}) - \widehat{D}(\rho_B||\sigma_B)\,.$$

$$H(\sigma_{AB}) := \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB}\,.$$

Weak QF BS-entropy (Bluhm-C.-Pérez Hernández '21)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following inequality holds whenever $\|H(\sigma_{AB})\|_\infty < 1/2$:

$$\widehat{D}(\rho_{AB}||\sigma_{AB}) \leq M(\sigma_{AB})\left[\widehat{D}_A(\rho_{AB}||\sigma_{AB}) + \widehat{D}_B(\rho_{AB}||\sigma_{AB})\right] + L(\rho_{AB}, \sigma_{AB}),$$

where $$M(\sigma_{AB}) := \frac{1}{1 - 2\|H(\sigma_{AB})\|_\infty}\,,$$

and $$L(\rho_{AB}, \sigma_{AB}) \leq f\left(\left\|\left[\rho_A^{1/2}, \sigma_A^{-1/2}\right]\right\|_\infty, \left\|\left[\rho_B^{1/2}, \sigma_B^{-1/2}\right]\right\|_\infty\right).$$

Consequences and possible applications

Consequences:

- Our results are equivalent to a violation of the property of **superadditivity** for the BS-entropy.

Possible applications:

- In **cryptography**, for entropy accumulation purposes.
- In **many-body systems**, to prove positivity of modified logarithmic Sobolev inequalities.

For further information, see **arXiv:2101.10312** and our long talk.

Thank you for your attention!