Bishop-Phelps problem

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Notation

- X, Y Banach spaces
- $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ base field
- $\mathbb{B}_X := \{x \in X \ : \ \|x\| \leq 1\}$
- $\mathbb{S}_X := \{x \in X \ : \ \|x\| = 1\}$
- X^* dual space of X
- $L(X,Y) := \{T : X \to Y \mid T$ linear and bounded$\}$
- $K(X,Y) := \{T \in L(X,Y) \mid T$ compact$\}$
- $F(X,Y) := \{T \in L(X,Y) \mid T$ with finite rank$\}$
- If $X = Y$, $L(X)$, $K(X)$, $F(X)$
- $F(X,Y) \subseteq K(X,Y) \subseteq L(X,Y)$

Norm-attaining functional

X (real or complex) Banach space, X^* dual of X, $x^* \in X^*$.

$$\|x^*\| := \sup \{|x^*(x)| \, : \, x \in \mathbb{B}_X\}$$

x^* **attains its norm** when this supremum is a maximum, i.e.,

$$\exists x_0 \in \mathbb{S}_X \, : \, |x^*(x_0)| = \|x^*\|$$

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Examples of norm-attaining functionals

- **In ℓ_1 :**

$$\begin{array}{rcccl} f: & \ell_1 & \longrightarrow & \mathbb{K} & \\ & x=\{x_n\} & \mapsto & f(x) & = \sum_{k=1}^{\infty} \frac{x_k}{k} \end{array}$$

For every $x \in \ell_1$,

$$|f(x)| \leq \sum_{k=1}^{\infty} \left|\frac{x_k}{k}\right| \leq \sum_{k=1}^{\infty} |x_k| < \infty \qquad (x \in \ell_1)$$

And if $x \in \mathbb{B}_{\ell_1}$, $|f(x)| \leq 1$, so $\|f\| \leq 1$.

Taking $x = (1, 0, \ldots)$, $\|x\|_1 = 1$ and $f(x) = 1 = \|f\|$. Thus, f attains its norm.

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- **In c_0:**

$$\begin{array}{rccll} g: & c_0 & \longrightarrow & \mathbb{K} & \\ & x = \{x_n\} & \mapsto & g(x) & = x_1 + x_2 \end{array}$$

For every $x \in c_0$, $\|x\|_\infty < \infty$, so $|g(x)| \leq 2\|x\|_\infty < \infty$, and $\|g\| \leq 2$.

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$$NA(X,\mathbb{K}) = \{x^* \in X^* \ : \ x^* \text{ attains its norm}\}$$

Hahn-Banach Theorem

Let X be a normed space over $\mathbb{K}$ and M a subspace. Let $g : M \to \mathbb{K}$ be continuous and linear. Then, there exists an extension $f : X \to \mathbb{K}$ of g, which is also linear and continuous, such that $\|f\| = \|g\|$.

Corollary

For every $x \in X$, there exists $f \in X^*$ verifying $\|f\| = 1$ and $f(x) = \|x\|$.

Given $x_0 \in \mathbb{S}_X$ there exists $f \in X^*$ with $\|f\| = 1$ and $f(x_0) = \|x_0\| = 1$.

$$\Rightarrow NA(X,\mathbb{K}) \neq \emptyset$$

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Examples of non norm-attaining functionals

- **In ℓ_1:**

$$\begin{array}{rcl} f: \quad \ell_1 & \longrightarrow & \mathbb{K} \\ x = \{x_n\} & \mapsto & f(x) = \sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right) x_k \end{array}$$

$$|f(x)| \leq \sum_{k=1}^{\infty} \left| \left(1 - \frac{1}{k}\right) x_k \right| \leq \sum_{k=1}^{\infty} |x_k| < \infty \qquad (x \in \ell_1)$$

Then, $\|f\| \leq 1$. Considering $e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots) \in \ell_1$, we have $\|e_n\| = 1 \ \forall n \in \mathbb{N}$ and

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$$x \in \mathbb{B}_{\ell_1} \Rightarrow |f(x)| < \|x\|_1 \leq 1$$

Then, f does not attain its norm.

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Reflexive space

X Banach space, X^{**} its bidual space

$$x \in X, \qquad J(x) : X^* \to \mathbb{K}$$

$$J(x)(f) = f(x) \qquad f \in X^*$$

A Banach space is **reflexive** when J is surjective.

James Theorem

A Banach space X is reflexive if, and only if, every continuous linear functional on X attains its norm on $\mathbb{B}_X$.

- X reflexive $\Rightarrow NA(X,\mathbb{K}) = L(X,\mathbb{K})$
- X non-reflexive $\Rightarrow NA(X,\mathbb{K}) \neq L(X,\mathbb{K})$

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Bishop-Phelps Theorem, Bull. AMS 1961

The set of norm-attaining functionals is dense in X^* (for the norm topology).

$$\overline{NA(X,\mathbb{K})} = L(X,\mathbb{K})$$

Norm-attaining operator

X, Y Banach spaces, $T \in L(X, Y)$

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T **attains its norm** when this supremum is a maximum, i.e.,

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Example of norm-attaining operator

$$\begin{aligned} T: \quad L^1(\mathbb{T}) &\longrightarrow c_0 \\ f &\mapsto \left\{\hat{f}(n)\right\} \end{aligned}$$

$$\begin{aligned} \|T\| &= \sup_{f\in\mathbb{S}_{L^1(\mathbb{T})}} \|T(f)\| = \sup_{f\in\mathbb{S}_{L^1(\mathbb{T})}} \left\|\left\{\hat{f}(n)\right\}\right\| \\ &= \sup_{f\in\mathbb{S}_{L^1(\mathbb{T})}} \sup_{n\in\mathbb{N}} \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} f(t)e^{-int}dt\right| \\ &\leq \sup_{f\in\mathbb{S}_{L^1(\mathbb{T})}} \sup_{n\in\mathbb{N}} \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|\,dt = \sup_{f\in\mathbb{S}_{L^1(\mathbb{T})}} \|f\| = 1 \end{aligned}$$

Fix $n_0 \in \mathbb{N}$ and consider $f(t) = e^{in_0t}$. Then,

$$\|T(f)\| = \sup_{n\in\mathbb{N}} \left|\hat{f}(n)\right| \geq \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in_0t}dt\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dt\right| = 1$$

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Fix $n_0 \in \mathbb{N}$ and consider $f(t) = e^{in_0t}$. Then,

$$\|T(f)\| = \sup_{n\in\mathbb{N}} \left|\hat{f}(n)\right| \ge \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-in_0t}dt\right| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dt\right| = 1$$

Example of norm-attaining operator

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Example of non norm-attaining operator

Consider $c = \{c_n\} \in \ell_\infty$ such that $|c_n| < \sup |c_n|$ and $T : \ell_2 \to \ell_2$ given by

$$T(x) = cx = \{c_n x_n\}, \qquad \forall x = \{x_n\} \in \ell_2$$

For any $x = \{x_n\} \in \ell_2$, we have

$$\|Tx\|^2 = \sum_{n\in\mathbb{N}} |c_n x_n|^2 < \sum_{n\in\mathbb{N}} (\sup |c_n|^2) |x_n|^2 = (\sup |c_n|^2) \|x\|^2$$

Therefore, $\|T\| \leq \sup |c_n|$. If we choose $x = e_n = \{\delta_{k,n}\}_{k\in\mathbb{N}}$, we have

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Let X be a finite dimensional space and Y arbitrary. Then,

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Corollary

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where $\{e_n\}$ is the sequence whose n-th term is 1 and the others are 0, and $\langle\cdot,\cdot\rangle$ is the scalar product of ℓ^2. The norm of T is 1, but for $x \neq 0$,

$$\|Tx\|^2 = \sum_{n\geq 1}\left(1-\frac{1}{n}\right)^2 |\langle x, e_n\rangle|^2 < \sum_{n\geq 1}|\langle x, e_n\rangle|^2 = \|x\|^2\ ,$$

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A Banach space X is reflexive if, and only if, for every finite dimensional Y, every $T \in L(X,Y)$ attains its norm.

Proposition

If X and Y verify

$$NA(X,Y) = L(X,Y)$$

then X is reflexive.

- X reflexive, Y finite dimensional (in particular, $\mathbb{K}$) $\Rightarrow NA(X,Y) = L(X,Y)$
- X finite dimensional $\Rightarrow NA(X,Y) = L(X,Y) \;\; \forall Y$
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Theorem

Let X and Y be two classical Banach spaces, i.e., they are of the form $L^p(\mu)$ or $C(S)$. Then, $NA(X,Y) = L(X,Y)$ if and only if $X = L^p(\mu)$, $Y = L^r(\nu)$, with $1 \leq r < p < \infty$ and one of the following holds

(a) $1 < r$ and μ and ν are atomic.

(b) $1 < r < 2$ and ν is atomic.

(c) $p > 2$, $r > 1$ and μ is atomic.

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The answer, in general, is negative.

Lindenstrauss' counterexample

$X = c_0$, Y strictly convex

$$T \in NA(c_0,Y) \Rightarrow T \in F(c_0,Y)$$

If there exists a non-compact operator from c_0 to Y, then $\overline{NA(c_0,Y)} \neq L(c_0,Y)$

Problem

$$\overline{NA(X)} = L(X) \text{ ?}$$

If Y is strictly convex and isomorphic to c_0, $X = c_0 \oplus_\infty Y$

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Properties A and B

Properties A and B

X has **property** A if $\overline{NA(X,Y)} = L(X,Y) \quad \forall Y$

Y has **property** B if $\overline{NA(X,Y)} = L(X,Y) \quad \forall X$

Lindenstrauss-Zizler Theorem

Lind.: $\overline{\{T \in L(X,Y) : T^{**} \in NA(X^{**},Y^{**})\}} = L(X,Y) \quad \forall X,Y$

Zizler: $\overline{\{T \in L(X,Y) : T^{*} \in NA(Y^{*},X^{*})\}} = L(X,Y) \quad \forall X,Y$

$\Rightarrow$ Every reflexive Banach space has property A.

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Examples

- $\mathbb{K}$ has property B.
- X finite dimensional has property A.
- X reflexive has property A.
- c_0 does not have property A.
- If Y strictly convex and there exists a non-compact operator from c_0 to Y, then Y does not have property B.

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$$\{(x_\lambda, x^*_\lambda) : \lambda \in \Lambda\} \subset \mathbb{S}_X \times \mathbb{S}_{X^*}, 0 \le \rho < 1$$

(1) $x^*_\lambda(x_\lambda) = 1 \qquad \forall \lambda \in \Lambda$

(2) $\lambda, \mu \in \Lambda,\ \lambda \neq \mu \Rightarrow |x^*_\lambda(x_\mu)| \le \rho$

(3α) $\|x^*\| = \sup\{|x^*(x_\lambda)| : \lambda \in \Lambda\} \qquad \forall x^* \in X^* \qquad$ (ej: ℓ_1)

(3β) $\|x\| = \sup\{|x^*_\lambda(x)| : \lambda \in \Lambda\} \qquad \forall x \in X \qquad$ (ej: c_0, ℓ_∞)

Lindenstrauss / Schachermayer

$$\beta \Rightarrow B$$

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Partington Theorem

Every Banach space can be renormed with β.

Schachermayer Theorem

Every WCG Banach space can be renormed with α.

Godun-Troyanski Theorem

Every Banach space X admitting a biorthogonal system with cardinality equal to dens$X$ can be renormed with α.

Remark

Not every Banach space can be renormed with property B. Indeed, there exists K Hausdorff compact topological space such that $C(K)$ cannot be renormed with property α.

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Relation with the Radon-Nikodym property

Dentability

X Banach space, C subset of X,
C is **dentable** if, for every $\varepsilon > 0$, we can find $x \in C$ such that $x \notin \overline{\text{co}}(C \setminus (x + \varepsilon \mathbb{B}_X))$.

Radon-Nikodym property

A Banach space X **has the Radon-Nikodym property (RNP)** if the Radon-Nikodym theorem holds for X-valued vector measures (w.r.t. every finite positive measure).

Theorem (Rieffel, Maynard, Huff, David, Phelps)

- X has the RNP if, and only if, every bounded subset of X is dentable.
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$RNP \Rightarrow A$ (for every equivalent norm)

Huff Theorem

X no $RNP \Rightarrow \exists X_1 \sim X \sim X_2 : \overline{NA(X_1, X_2)} \neq L(X_1, X_2)$

Non-linear optimization principle of Bourgain-Stegall

$RNP \Leftarrow A$ (for every equivalent norm)

B (for every equivalent norm) $\Rightarrow RNP$

The reciprocal is not true.

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Conjecture

RNP $\Leftrightarrow \overline{NA(X)} = L(X)$ for every equivalent norm

Proposition

Y Banach space, $X \cong Y \oplus Y$

$X \cong Y \oplus_1 Y \;\Rightarrow\; \|x\|_X = \|y_1\|_Y + \|y_2\|_Y \;\; \forall x = (y_1, y_2)$

$X \cong Y \oplus_\infty Y \;\Rightarrow\; \|x\|_X = \max\{\|y_1\|_Y, \|y_2\|_Y\} \;\; \forall x = (y_1, y_2)$

X verifies $\overline{NA(X)} = L(X)$ for every equivalent norm, if, and only if, X has the RNP.

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Counterexamples

Gowers' counterexample

No infinite dimensional Hilbert space has property B.

For $1 < p < \infty$, ℓ_p and L_p do not have property B.

Acosta's counterexample

No infinite dimensional strictly convex Banach space has property B.

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Open problems

- Do finite dimensional spaces have property B?
  In particular, does $\mathbb{R}^2$, with the euclidean norm, have property B?
- Characterize the compacts K such that $C(K)$ has property B.
- RNP $\Leftrightarrow \overline{NA(X)} = L(X)$ for every equivalent norm ?

Norm-attaining compact operators

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$$NAK(X,Y) := K(X,Y) \cap NA(X,Y)$$

Problem

$$\overline{NAK(X,Y)} = K(X,Y)?$$

Remark

If X is reflexive, $NAK(X,Y) = K(X,Y)\ \forall Y$.

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Approximation property

X has the **approximation property (AP)** if for every compact $K \subset X$ and every $\varepsilon > 0$ there exists an operator $T \in F(X)$ such that $\|Tx - x\| < \varepsilon$ for every $x \in K$.

Grothendieck Theorem

- Y has AP $\Leftrightarrow \overline{F(Z,Y)} = K(Z,Y)\ \forall Z$.
- X^* has AP $\Leftrightarrow \overline{F(X,Z)} = K(X,Z)\ \forall Z$.
- X^* has AP $\Rightarrow X$ has AP.

Grothendieck Lemma

A Banach space Y has the approximation property if, and only if, $\overline{F(X,Y)} = K(X,Y)$ for every closed subspace X of c_0.

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Examples

- (Enflo) There exists $X \leqslant c_0$ without AP.
- (Davie) There exists $X \leqslant \ell_p$ without AP for $1 \le p < 2$.
- (Szankowski) There exists $X \leqslant \ell_p$ without AP for $2 < p < \infty$.

Martín, '14

There exist compact linear operators between Banach spaces which cannot be approximated by norm-attaining operators.

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$$¿\overline{NAK(X,Y)} = K(X,Y)?$$

The answer is negative, in general.

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Properties AK and BK

- X has property AK if $\overline{NAK(X,Y)} = K(X,Y) \ \forall Y$.
- Y has property BK if $\overline{NAK(X,Y)} = K(X,Y) \ \forall X$.

Examples

- Finite-dimensional spaces have property AK.
- $Y = \mathbb{K}$ has property BK.
- Real finite-dimensional polyhedral spaces have property BK.

Example

There exists $X \leqslant c_0$ failing property AK and Y failing BK.

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Thank you!