Approximate tensorization of the relative entropy for noncommuting conditional expectations

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Objective

We derive a new generalisation of the **strong subadditivity** of the entropy to the setting of general conditional expectations onto arbitrary finite-dimensional von Neumann algebras. This inequality is called **approximate tensorization of the relative entropy** and, in particular, constitutes a key step in modern proofs of **logarithmic Sobolev inequalities** for classical and quantum lattice spin systems.

Introduction

1. A fundamental property of entropy is the **strong subadditivity** inequality (SSA): Given $\mathcal{H}_{ABC} := \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, and a state ρ_{ABC} on $\mathcal{H}_{ABC}$, the following holds:

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})\,, \qquad \text{(SSA)}$$

where $S(\rho) = -\operatorname{tr}[\rho \ln \rho]$ is the von Neumann entropy and for any subsystem D of ABC, $\rho_D := \operatorname{tr}_{D^c}[\rho_{ABC}]$ denotes the marginal state on D.

2. In terms of the **relative entropy** $D(\rho||\sigma) := \operatorname{tr}[\rho\,(\ln \rho - \ln \sigma)]$, (SSA) takes the form

$$D\left(\rho_{ABC}\middle\|\rho_B \otimes \mathbb{1}_{AC}/d_{\mathcal{H}_{AC}}\right) \leq D\left(\rho_{ABC}\middle\|\rho_{AB} \otimes \mathbb{1}_C/d_{\mathcal{H}_C}\right) + D\left(\rho_{ABC}\middle\|\rho_{BC} \otimes \mathbb{1}_A/d_{\mathcal{H}_A}\right)\,.$$

3. For finite-dimensional **von Neumann algebras** $\mathcal{M} \subset \mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{N}$, let $E^{\mathcal{M}}, E_1, E_2$ be their corresponding conditional expectations. When $E_1 \circ E_2 = E_2 \circ E_1 = E^{\mathcal{M}}$:

$$D(\rho||E^{\mathcal{M}}_*(\rho)) \leq D(\rho||E_{1*}(\rho)) + D(\rho||E_{2*}(\rho))\,,$$

where the *coarse-graining maps* $E^{\mathcal{M}}_*, E_{1*}, E_{2*}$ are the Hilbert-Schmidt duals of $E^{\mathcal{M}}, E_1, E_2$. To recover (SSA) take $\mathcal{N} \equiv \mathcal{B}(\mathcal{H}_{ABC})$, $\mathcal{N}_{1,2} \equiv \mathcal{B}(\mathcal{H}_{AB,BC})$ and $\mathcal{M} \equiv \mathcal{B}(\mathcal{H}_B)$.

4. For $\mathcal{M} \equiv \mathbb{C}\mathbb{1}_{\mathcal{H}}$, in the classical [5] and quantum setting [2, 3] the following inequality is called **approximate tensorization of the relative entropy**:

$$D(\rho||\sigma) \leq \frac{1}{1-2c}\left(D(\rho||E_{1*}(\rho)) + D(\rho||E_{2*}(\rho))\right)\,,$$

where $\sigma := E^{\mathcal{M}}_*(\rho)$ and c is measures the distance from the commuting assumption. Typically, $c = 0$ at infinite temperature, and remains small for conditional expectations onto far apart regions and at high enough temperature.

Approximate tensorization

Here [1] we take one step further and prove a **weak approximate tensorization** for the relative entropy, which amounts to the existence of positive constants $c \geq 1$ and $d \geq 0$ such that

$$D(\rho||E^{\mathcal{M}}_*(\rho)) \leq c\left(D(\rho||E_{1*}(\rho)) + D(\rho||E_{2*}(\rho))\right) + d\,. \qquad \text{(AT(c, d))}$$

We estimate c and d in terms of the interactions appearing in the Hamiltonian of quantum lattice spin systems.

First results on AT

Weak approximate tensorization for conditional expectations

$$D(\rho||\rho_{\mathcal{M}}) \leq D(\rho||\rho_1) + D(\rho||\rho_2) + d\,.$$

where $\rho_i := E_{i*}(\rho)$ for $i = 1, 2$, $\rho_{\mathcal{M}} := E^{\mathcal{M}}_*(\rho)$ and the constant d is given by

$$d := \sup_{\rho \in \mathcal{D}(\mathcal{N}_2)} \inf\left\{\ln(\lambda) \middle| E_{1*}(\rho) \leq \lambda\eta \text{ for some } \eta \in \mathcal{D}(\mathcal{M})\right\}.$$

Under a noncommutative change of measure argument

Assume that $\mathrm{AT}(1, d)$ holds at infinite temperature for the constant d of the previous result. Then, $\mathrm{AT}(c, d')$ with $c = \frac{\lambda_{\max}(\sigma)}{\lambda_{\min}(\sigma)}$ and $d' = \lambda_{\max}(\sigma)\, d_{\mathcal{H}}\, d$ holds:

$$D(\rho||E^{\mathcal{M}}_*(\rho)) \leq \frac{\lambda_{\max}(\sigma)}{\lambda_{\min}(\sigma)}\left(D(\rho||E_{1*}(\rho)) + D(\rho||E_{2*}(\rho))\right) + \lambda_{\max}(\sigma)\, d_{\mathcal{H}}\, d\,.$$

Conditional expectations

Heat-bath dynamics: Given σ a faithful density matrix on the finite-dimensional algebra $\mathcal{N}$, $\mathcal{M} \subset \mathcal{N}$ a subalgebra and denoting by E_τ the conditional expectation onto $\mathcal{M}$ with respect to the MMS, the **Heat-bath conditional expectation** is given by:

$$E_\sigma := \lim_{n \to \infty} \mathcal{A}_\sigma^n\,, \qquad \text{for } \mathcal{A}_\sigma(X) := E_\tau(\sigma)^{-\frac{1}{2}}\, E_\tau[\sigma^{\frac{1}{2}}\, X\, \sigma^{\frac{1}{2}}]\, E_\tau(\sigma)^{-\frac{1}{2}}\,.$$

Davies dynamics: For $\mathcal{L}^{\mathrm{D},\beta}$ the generator of a QMS modelling the evolution of a system weakly-coupled to a heat bath, the **Davies conditional expectation** is given by:

$$E^{\mathrm{D},\beta} := \lim_{t \to \infty} \mathrm{e}^{t\mathcal{L}^{\mathrm{D},\beta}}\,,$$

Both conditional expectations above are equal.

Main result

As a subalgebra of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H} = \bigoplus_{i \in I_{\mathcal{M}}} \mathcal{H}_i \otimes \mathcal{K}_i$, $\mathcal{M}$ bears the following block diagonal decomposition:

$$\mathcal{M} \equiv \bigoplus_{i \in I_{\mathcal{M}}} \mathcal{B}(\mathcal{H}_i) \otimes \mathbb{1}_{\mathcal{K}_i}\,, \qquad \text{so that} \qquad \forall \rho \in \mathcal{D}(\mathcal{N})\,,\ \rho_{\mathcal{M}} := \sum_{i \in I_{\mathcal{M}}} \operatorname{tr}_{\mathcal{K}_i}[P_i \rho P_i] \otimes \tau_i\,,$$

where P_i is the projection onto the i-th diagonal block and τ_i is a full-rank state on $\mathcal{K}_i$.

Now, we wish to compare the state ρ with a **classical-quantum state** according to the decomposition given by $\mathcal{M}$. To this end we introduce the **Pinching map** with respect to the $\mathcal{H}_i$ in the decomposition of $\mathcal{M}$, denoted by $\mathcal{P}_{\rho_{\mathcal{M}}}$.

Approximate tensorization via Pinching

Define

$$c_1 := \max_{i \in I_{\mathcal{M}}} \left\| E_1^{(i)} \circ E_2^{(i)} - (E^{\mathcal{M}})^{(i)} : \mathbb{L}_1(\tau_i) \to \mathbb{L}_\infty \right\|\,,$$

where the $E_1^{(i)}$ are defined in each $\mathcal{H}_i$. Then, $\mathrm{AT}(c, d)$ holds for the following constants:

$$c := \frac{1}{(1-2c_1)}, \qquad d := D(\rho||\mathcal{P}_{\rho_{\mathcal{M}}}(\rho))\,.$$

Application: Pinching onto different bases

We take $\mathcal{H} = \mathbb{C}^l$, $\mathcal{M} \equiv \mathbb{C}\mathbb{1}_\ell$ and assume that $\mathcal{N}_1$, resp. $\mathcal{N}_2$, is the diagonal onto some orthonormal basis $|e_k^{(1)}\rangle$, resp. $|e_k^{(2)}\rangle$. Hence for each $i \in \{1, 2\}$, E_i denotes the Pinching map onto the diagonal span $\left\{|e_k^{(i)}\rangle\langle e_k^{(i)}|\right\}$ and $E^{\mathcal{M}} = \frac{1}{\ell}\operatorname{tr}[\cdot]$. Then, $\mathrm{AT}((1-2\varepsilon)^{-1}, 0)$ holds:

$$D(\rho||\ell^{-1}\mathbb{1}) \leq \frac{1}{1-2\varepsilon}\left(D(\rho||E_{1*}(\rho)) + D(\rho||E_{2*}(\rho))\right).$$

This implies that the primitive quantum Markov semigroup $\mathrm{e}^{t\mathcal{L}}$ satisfies a **modified logarithmic Sobolev inequality** of constant $1 - 2\varepsilon$, where

$$\mathcal{L}(X) := E_1(X) + E_2(X) - 2X\,.$$

Application: Entropic uncertainty relations

Strengthened entropic uncertainty relation

Given a finite alphabet $\mathcal{Z} \in \{\mathcal{X}, \mathcal{Y}\}$, let $E_{\mathcal{Z}}$ denote the Pinching channels onto the orthonormal basis $\{|e_z^{(\mathcal{Z})}\rangle\}_{z \in \mathcal{Z}}$ corresponding to the measurement **Z**. Assume further that $c_1 = d_A \max_{x,y} \left||\langle e_x^{(\mathcal{X})}|e_y^{(\mathcal{Y})}\rangle|^2 - \frac{1}{d_A}\right| < \frac{1}{2}$. Then the following strenghtened entropic uncertainty relation holds for any state $\rho \in \mathcal{D}(\mathcal{H}_A)$:

$$S(X)_{E_{\mathcal{X}}(\rho)} + S(Y)_{E_{\mathcal{Y}}(\rho)} \geq (1 + 2c_1)\, S(A)_\rho + (1 - 2c_1) \ln d_A\,.$$

Main application: MLSI

In [4], a result of AT(c,0) derived from the ones presented here is the key tool to prove positivity of the MLSI for quantum spin systems.

MLSI for quantum spin systems

Given a 2-local potential over $\mathbb{Z}^d$, some finite region $\Lambda \subset \mathbb{Z}^d$ and $\beta > 0$, and the Gibbs state $\sigma^\Lambda = e^{-\beta H_\Lambda}/\operatorname{tr}\left[e^{-\beta H_\Lambda}\right]$, assume that for every Λ there is a local, primitive, reversible and frustration-free Lindbladian $\mathcal{L}_\Lambda$ such that its local terms satisfy a **complete modified logarithmic Sobolev inequality** and that it satisfies a suitable clustering of correlations condition (which implies a result of **strong approximate tensorization**). Then, there is a constant $\alpha > 0$ independent of $|\Lambda|$ such that for every initial state ρ

$$D(e^{t\mathcal{L}_{\Lambda*}}(\rho)||\sigma^\Lambda) \leq e^{-\alpha t} D(\rho||\sigma^\Lambda)\,,$$

References

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