Continuity of quantum entropic quantities via almost convexity

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Objects

- $\mathcal{H}$ is a finite-dimensional Hilbert space.
- $\mathcal{H}_{AB}$ (resp. $\mathcal{H}_{ABC}$) is a bi-partite (resp. tri-partite) space.
- $\mathcal{S}(\mathcal{H})$ is the set of density matrices.

Entropies and derived quantities:

von Neumann entropy	$S(\rho) := -\operatorname{tr}[\rho \log \rho]$
Relative entropies	Umegaki relative entropy $D(\rho\|\sigma) := \operatorname{tr}[\rho(\log \rho - \log \sigma)]$ Belavkin-Staszewski entropy $\widehat{D}(\rho\|\sigma) := \operatorname{tr}\left[\rho \log\left(\rho^{1/2}\sigma^{-1}\rho^{1/2}\right)\right]$
Conditional entropy	$H_\rho(A\|B) := S(\rho_{AB}) - S(\rho_B) \geq -\log d_A$
Mutual information	$I_\rho(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \geq 0$
Conditional mutual information	$I_\rho(A:C\|B) = I_\rho(A:BC) - I_\rho(A:B) \geq 0$

Problem

Divergence

A **divergence** is a function $\mathbb{D} : \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \times \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B) \to [0, +\infty)$ that satisfies the data processing inequality, i.e.

$$\mathbb{D}(\rho_{AB} \| \sigma_{AB}) \geq \mathbb{D}(\operatorname{tr}_A[\rho_{AB}] \| \operatorname{tr}_A[\sigma_{AB}]) = \mathbb{D}(\rho_B \| \sigma_B) \, .$$

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How to provide continuity bounds?

$$|\mathbb{D}(\rho_1 \| \sigma_1) - \mathbb{D}(\rho_2 \| \sigma_2)| \leq f(\|\rho_1 - \rho_2\|_1, \|\sigma_1 - \sigma_2\|_1)$$

with $\|\cdot\|_1 := \operatorname{tr}[| \cdot |]$.

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Example: Conditional entropy (Winter, '16)

Continuity bound for the conditional entropy

Given $\varepsilon > 0$, consider $\rho^1_{AB}, \rho^2_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ with $\frac{1}{2}\left\|\rho^1 - \rho^2\right\|_1 \leq \varepsilon$. Then, we have

$$|H_{\rho^1}(A|B) - H_{\rho^2}(A|B)| \leq 2\varepsilon \log d_A + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right),$$

with $h\left(\frac{1}{1+\varepsilon}\right) = -\frac{\varepsilon}{1+\varepsilon}\log\frac{1}{1+\varepsilon} - \frac{\varepsilon}{1+\varepsilon}\log\frac{\varepsilon}{1+\varepsilon}$ the binary entropy.

Proof:

- Recall that $H_\omega(A|B) = S(\omega_{AB}) - S(\omega_B)$, $\forall \omega \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

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$$D(\rho \| \sigma) := \begin{cases} \operatorname{tr}[\rho \log \rho - \rho \log \sigma] & \text{if } \ker \sigma \subseteq \ker \rho \,, \\ +\infty & \text{else} \,. \end{cases}$$

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The ALAFF method

Prior work: Alicki-Fannes '04, Winter '16, Shirokov '20.

ALAFF function

A function $f : \mathcal{S}(\mathcal{H}) \to [0, +\infty)$ is **almost locally affine** if for every $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ and $p \in [0, 1]$, we have

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s-perturbed Δ-invariant subsets

Let $s \in [0, 1)$. A subset $\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{H})$ is called *s-perturbed Δ-invariant*, if for $\rho, \sigma \in \mathcal{S}_0$ with $\rho \neq \sigma$ there exists $\tau \in \mathcal{S}(\mathcal{H})$ such that the two states

$$\Delta^{\pm}(\rho, \sigma, \tau) = s\tau + (1-s)\varepsilon^{-1}[\rho - \sigma]_{\pm}$$

lie again in $\mathcal{S}_0$. Here $\varepsilon := \frac{1}{2}\|\rho - \sigma\|_1$ and $[A]_{\pm}$ denotes the negative and positive part of a self-adjoint operator, respectively.

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Theorem. Almost locally affine (ALAFF) method

$\mathcal{S}_0 \subseteq \mathcal{S}(\mathcal{H})$ be a s-perturbed Δ-invariant convex subset of $\mathcal{S}(\mathcal{H})$ containing more than one element, f an ALAFF function. Then f is uniformly continuous if

$$C_f^s := \sup_{\substack{\rho,\sigma \in \mathcal{S}_0 \\ \frac{1}{2}\|\rho - \sigma\|_1 = 1-s}} |f(\rho) - f(\sigma)| < +\infty\,.$$

In this case, for $\varepsilon \in (0, 1]$

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Comparison to conditional entropy case

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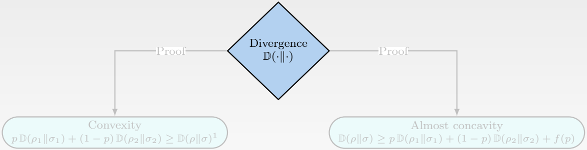
From almost convexity to continuity bounds



→ Crucial: "Well-behaved" remainder function.

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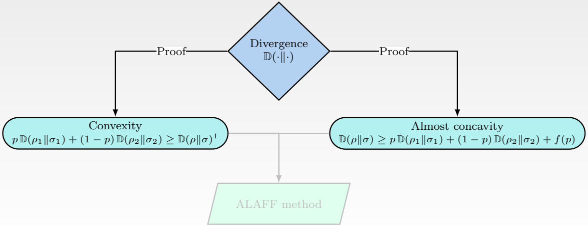

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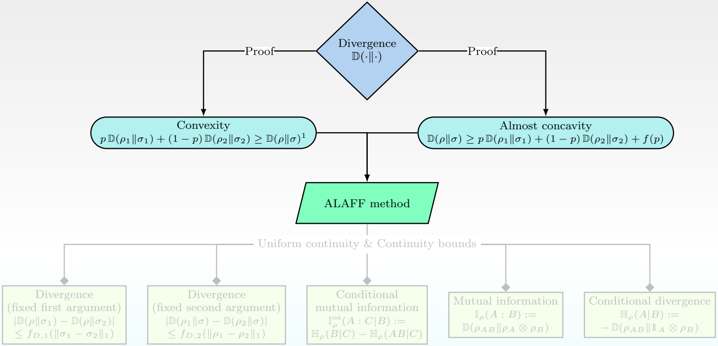

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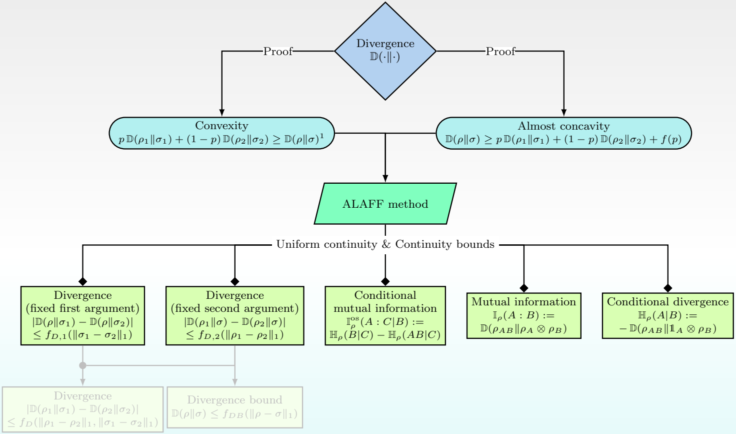

→ Crucial: "Well-behaved" remainder function.

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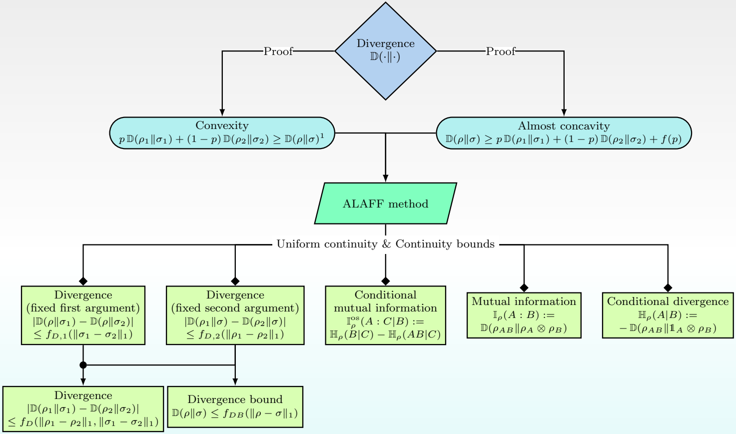

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Almost concavity of the relative entropy

Relative entropy

$$D(\rho\|\sigma) := \begin{cases} \operatorname{tr}[\rho\log\rho - \rho\log\sigma] & \text{if } \ker\sigma \subseteq \ker\rho \\ +\infty & \text{else} \end{cases}$$

Theorem. Almost concavity of the relative entropy

Let $(\rho_1, \sigma_1), (\rho_2, \sigma_2) \in \mathcal{S}_{\ker} := \{(\rho,\sigma) \in \mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}) \ : \ \ker\sigma \subseteq \ker\rho\}$ and $p \in [0,1]$. With $\rho = p\rho_1 + (1-p)\rho_2$ and $\sigma = p\sigma_1 + (1-p)\sigma_2$,

$$\boxed{D(\rho\|\sigma) \geq pD(\rho_1\|\sigma_1) + (1-p)D(\rho_2\|\sigma_2) - h(p)\frac{1}{2}\|\rho_1 - \rho_2\|_1 - f_{c_1,c_2}(p)}$$

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The constants in f_{c_1,c_2} are non-negative real numbers and are given by

$$c_j := \int_{-\infty}^{\infty} dt\beta_0(t)\operatorname{tr}\left[\rho_j\sigma_j^{\frac{it-1}{2}}\sigma_k\sigma_j^{\frac{-it-1}{2}}\right] < \infty, \qquad j,k = 1,2, \quad j \neq k,$$

with β_0 a probability density on $\mathbb{R}$.

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Remarks:

- The result is tight!

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Derived continuity and divergence bounds

Quantity	**Bound** $(\varepsilon \geq \frac{1}{2}\|\rho - \sigma\|_1)$
Conditional entropy	$\lvert H_\rho(A\vert B) - H_\sigma(A\vert B)\rvert \leq 2\varepsilon \log d_A + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$
Mutual information	$\lvert I_\rho(A:B) - I_\sigma(A:B)\rvert \leq 2\varepsilon \log \min\{d_A, d_B\} + 2(1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$
Conditional mutual information	$\lvert I_\rho(A:B\vert C) - I_\sigma(A:B\vert C)\rvert$ $\leq 2\varepsilon \log \min\{d_A, d_B\} + 2(1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$
Divergence bound	$D(\rho\\|\sigma) \leq \varepsilon \log \widetilde{m}_\sigma^{-1} + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$
CB (1st input)	$\lvert D(\rho_1\\|\sigma) - D(\rho_2\\|\sigma)\rvert \leq \varepsilon \log \widetilde{m}_\sigma^{-1} + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$
CB (2nd input)	$\lvert D(\rho\\|\sigma_1) - D(\rho\\|\sigma_2)\rvert \leq \frac{3\log^2 \widetilde{m}^{-1}}{1-\widetilde{m}} \sqrt{\varepsilon}$
Relative entropy	$\lvert D(\rho_1\\|\sigma_1) - D(\rho_2\\|\sigma_2)\rvert$ $\leq \left(1 + \frac{\log \widetilde{m}^{-1}}{\sqrt{2}}\right) \\|\rho_1 - \rho_2\\|_1^{1/2} + \frac{5\log^2 \widetilde{m}^{-1}}{\sqrt{2}(1-\widetilde{m})} \\|\sigma_1 - \sigma_2\\|_1^{1/2}$

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Previously known , Compare to previous bounds.

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- Condition for a state to be an *approximate quantum Markov chain.*
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- Continuity bound for the *Rains information.*
- Bound on the distance between BS and relative entropy.

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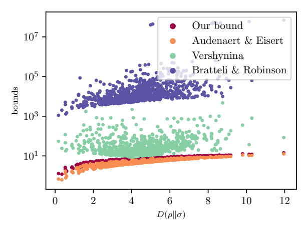
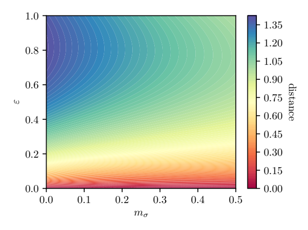
Divergence bounds





Bound by	not full rank ρ	not full rank σ	Bound on $D(\rho\|\sigma)$
Our bound	✓	✓	$\varepsilon \log \widetilde{m}_\sigma^{-1} + (1+\varepsilon) h\left(\frac{\varepsilon}{1+\varepsilon}\right)$
Audenaert Eisert '11	✓	x	$(m_\sigma + \varepsilon)\log\left(\frac{m_\sigma+\varepsilon}{m_\sigma}\right) - m_\rho \log\left(\frac{m_\rho+\varepsilon}{m_\rho}\right)$
Vershynina '19	x	x	$2\varepsilon\lambda_\rho \frac{\log m_\rho - \log m_\sigma}{m_\rho - m_\sigma}$
Bratteli Robinson '81	x	x	$m_\sigma^{-1}\|\rho-\sigma\|_\infty$

Table: Here $\varepsilon = \frac{1}{2}\|\rho-\sigma\|_1$ and $m_\cdot$ and $\widetilde{m}_\cdot$ are the minimal and the minimal non-zero eigenvalue of the quantum state in the index, respectively. Further λ_ρ is the maximal eigenvalue of ρ.

Almost concavity of the Belavkin-Staszewski entropy

Belavkin-Staszewski entropy

$$\widehat{D}(\rho\|\sigma) := \begin{cases} \operatorname{tr}\left[\rho \log\left(\rho^{1/2}\sigma^{-1}\rho^{1/2}\right)\right] & \text{if } \ker\sigma \subseteq \ker\rho \\ +\infty & \text{else} \end{cases}$$

Theorem. Almost concavity of the Belavkin-Staszewski entropy

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- However, the BS-conditional entropy might be discontinuous unless $\sigma > 0$. Dependence on $\hat{c}_0$?

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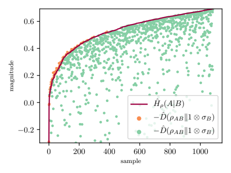
- If $\rho_i, \sigma_i \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $\sigma_i = (\rho_i)_A \otimes \mathbb{1}_B/d_B$ for $i = 1, 2$, then $\widehat{f}(p) = \hat{c}_0 h(p)$.
- When $[\rho_i, \sigma_i] = 0$, we have $D(\rho_i\|\sigma_i) = \widehat{D}(\rho_i\|\sigma_i)$. Thus, in that case we would expect $\widehat{f}(p) = f(p) = h(p)$.
- However, the BS-conditional entropy might be discontinuous unless $\sigma > 0$. Dependence on $\hat{c}_0$?

Taming the BS-entropy

- Definition of the *BS-conditional entropy*



$$\hat{H}_\rho(A|B) := -\widehat{D}(\rho_{AB}\|\mathbb{1}_A \otimes \rho_B)$$

Discontinuous on $\mathcal{S}(\mathcal{H})$

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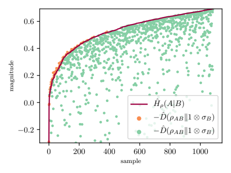

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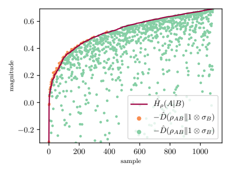

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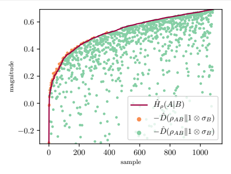

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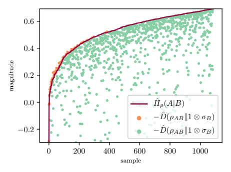

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$$\widehat{I}_\rho(A:B|C) = \widehat{H}_\rho(A|C) - \widehat{H}_\rho(A|BC)$$

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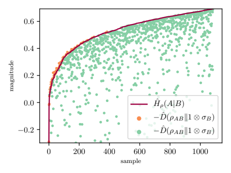

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Derived continuity and divergence bounds

Quantity	**Bound** $(\rho, \sigma \in \mathcal{S}_{\geq m}(\mathcal{H}),\ \varepsilon \geq \frac{1}{2}\|\rho - \sigma\|_1,\ l_m := 1 - m d_{\mathcal{H}})$	**Order**
BS-Cond Ent	$\lvert \widehat{H}_\rho(A\|B) - \widehat{H}_\sigma(A\|B)\rvert \leq 2 l_m^{-1} \varepsilon \log d_A + \frac{l_m+\varepsilon}{l_m}(f_{m^{-1},m^{-1}} + m^{-1} h)\left(\frac{\varepsilon}{l_m+\varepsilon}\right)$	$\sim m^{-1}\sqrt{\varepsilon}$
BS-MI	$\lvert \widehat{I}_\rho(A:B) - \widehat{I}_\sigma(A:B)\rvert \leq 2 l_m^{-1} \varepsilon (\log \min\{d_A, d_B\} + \log m^{-1}) + \frac{l_m+\varepsilon}{l_m} z_m\left(\frac{\varepsilon}{l_m+\varepsilon}\right)$	$\sim m^{-1}\sqrt{\varepsilon}$
BS-CMI	$\lvert \widehat{I}_\rho(A:B\|C) - \widehat{I}_\sigma(A:B\|C)\rvert \leq 2\,\varepsilon\, l_m^{-1} \log \min\{d_A, \sqrt{d_{ABC}}\} + 2 g_m(\varepsilon)$	$\sim m^{-1}\sqrt{\varepsilon}$
Div. bound	$\widehat{D}(\rho\|\sigma) \leq \varepsilon \log m_\sigma^{-1} + (1+\varepsilon) m_\sigma^{-1} h\left(\frac{\varepsilon}{1+\varepsilon}\right)$	$\sim m^{-1}\sqrt{\varepsilon}$

Applications:

- Continuity bound for the *variational BS-conditional entropy.*
- Continuity bound for the *BS-entropy of entanglement.*
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- Continuity bound for the *relative entropy of entanglement*.
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Approximate quantum Markov chain

$\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$ is a *quantum Markov chain* ($A \leftrightarrow B \leftrightarrow C$) iff

$$\rho_{ABC} = \rho_{AB}^{1/2} \rho_B^{-1/2} \rho_{BC} \rho_B^{-1/2} \rho_{AB}^{1/2} \, .$$

It is an *ε-approximate quantum Markov chain* iff

$$I_\rho(A : C|B) < \varepsilon \, .$$

Carlen-Vershynina '20 + Continuity bound for CMI:

$$\begin{aligned}\left(\frac{\pi}{8}\right)^4 \left\| \rho_B^{-1} \right\|_\infty^{-2} \left\| \rho_{ABC}^{-1} \right\|_\infty^{-2} \left\| \rho_{ABC} - \rho_{AB}^{1/2} \rho_B^{-1/2} \rho_{BC} \rho_B^{-1/2} \rho_{AB}^{1/2} \right\|_1^4 \\ \leq I_\rho(A : C|B) \\ \leq 2 \left(\log \min\{d_A, d_C\} + 1 \right) \left\| \rho_{ABC} - \rho_{AB}^{1/2} \rho_B^{-1/2} \rho_{BC} \rho_B^{-1/2} \rho_{AB}^{1/2} \right\|_1^{1/2}\end{aligned}$$

ρ_{ABC} is an *ε-approximate quantum Markov chain* iff

$$\rho_{ABC} \sim \rho_{AB}^{1/2} \rho_B^{-1/2} \rho_{BC} \rho_B^{-1/2} \rho_{AB}^{1/2} \, .$$

Approximate BS-recoverable state

$\rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC})$ is a *BS-recoverable state* (Bluhm-C. '20) iff

$$\rho_{ABC} = \rho_{AB}\rho_B^{-1}\rho_{BC} \, .$$

It is an ε*-approximate BS-recoverable state* iff

$$\widehat{I}_\rho(A:C|B) < \varepsilon \, .$$

Bluhm-C. '20 + Continuity bound for CMI (If valid for non-positive matrices!)

$$\begin{aligned}\left(\frac{\pi}{8}\right)^4 \left\|\rho_{ABC}^{-1/2}\rho_{AB}\otimes\rho_C\rho_{ABC}^{-1/2}\right\|_\infty^{-4}\left\|\rho_{ABC}^{-1}\right\|_\infty^{-2}\left\|\rho_{ABC}-\rho_{AB}\rho_B^{-1}\rho_{BC}\right\|_1^4 \\ \leq \widehat{I}_\rho(A:C|B) \\ \leq \mathcal{K}\left(\left\|\rho_{ABC}^{-1}\right\|_\infty, d_{ABC}\right)\left\|\rho_{ABC}-\rho_{AB}\rho_B^{-1}\rho_{BC}\right\|_1^{1/2}\end{aligned}$$

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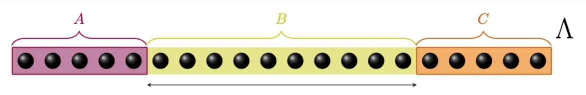

(Bluhm-C.-Perez Hernandez, '22)

If ρ_{ABC} is the Gibbs state of a local, translation-invariant Hamiltonian in 1D (at every temperature):

$$\left\| \rho_{ABC} - \rho_{AB}\rho_B^{-1}\rho_{BC} \right\|_1 \leq \delta(|B|) \,,$$

with δ decaying superexponentially fast.

- Thus, if the CB was valid for non-positive matrices,

$$\widehat{I}_\rho(A:C|B) \leq \mathcal{K}\left(\left\| \rho_{ABC}^{-1} \right\|_\infty, d_{ABC} \right) \left\| \rho_{ABC} - \rho_{AB}\rho_B^{-1}\rho_{BC} \right\|_1^{1/2} \,,$$

  and the BS-CMI would decay in 1D superexponentially fast at every temperature!
- For the CMI in 1D, it is only known exponentially fast at high temperature (Kuwahara-Kato-Brandao '20) and subexponentially fast at low temperature (Kato-Brandao '19).

Continuity of optimized quantities

Let $\mathcal{C} \subset \mathcal{S}(\mathcal{H})$ be a compact convex subset with at least one positive definite state. We can define the minimal distance to $\mathcal{C}$ in terms of the relative entropy (resp. BS-entropy) as

$$D_{\mathcal{C}}(\rho) := \inf_{\gamma \in \mathcal{C}} D(\rho \| \gamma) \quad \left(\widehat{D}_{\mathcal{C}}(\rho) := \inf_{\gamma \in \mathcal{C}} \widehat{D}(\rho \| \gamma) \right). \tag{1}$$

Then, this quantity is almost concave:

$$D_{\mathcal{C}}(p\rho_1 + (1-p)\rho_2) \geq pD_{\mathcal{C}}(\rho_1) + (1-p)D_{\mathcal{C}}(\rho_2) - h(p)\,.$$

(similar for the BS-entropy)

This outputs continuity bounds for

- The (BS-) relative entropy of entanglement.
- The (BS-) Rains information.
- The variational BS-conditional entropy.

Outlook & Future work

- Improve bound on BS-entropy
  - → In case ρ and σ commute, we would expect the bound of the BS and relative entropy to coincide.

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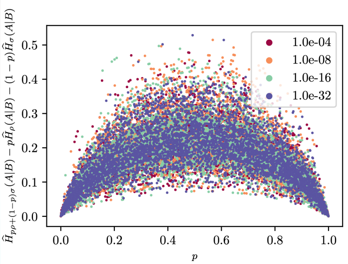

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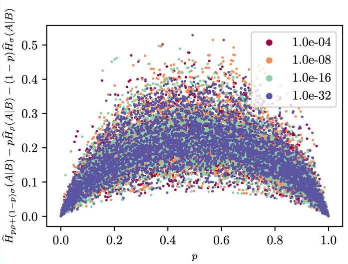

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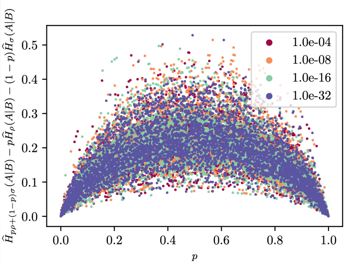

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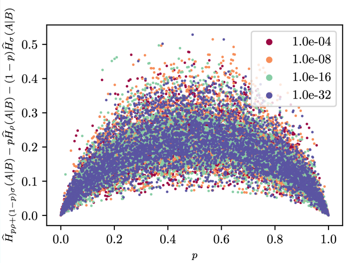

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Summary

- We have constructed the ALAFF Method to derive continuity bounds of entropic quantities.
- We have recovered best-known bounds for the relative entropy in addition to new divergence and continuity bounds.
- We have obtained first bounds on BS-conditional entropy, BS-mutual information, BS-conditional mutual information and divergence and continuity bounds for the BS.
- This yields numerous applications in quantum information theory.

Thank you for your attention!

Do you have any questions?