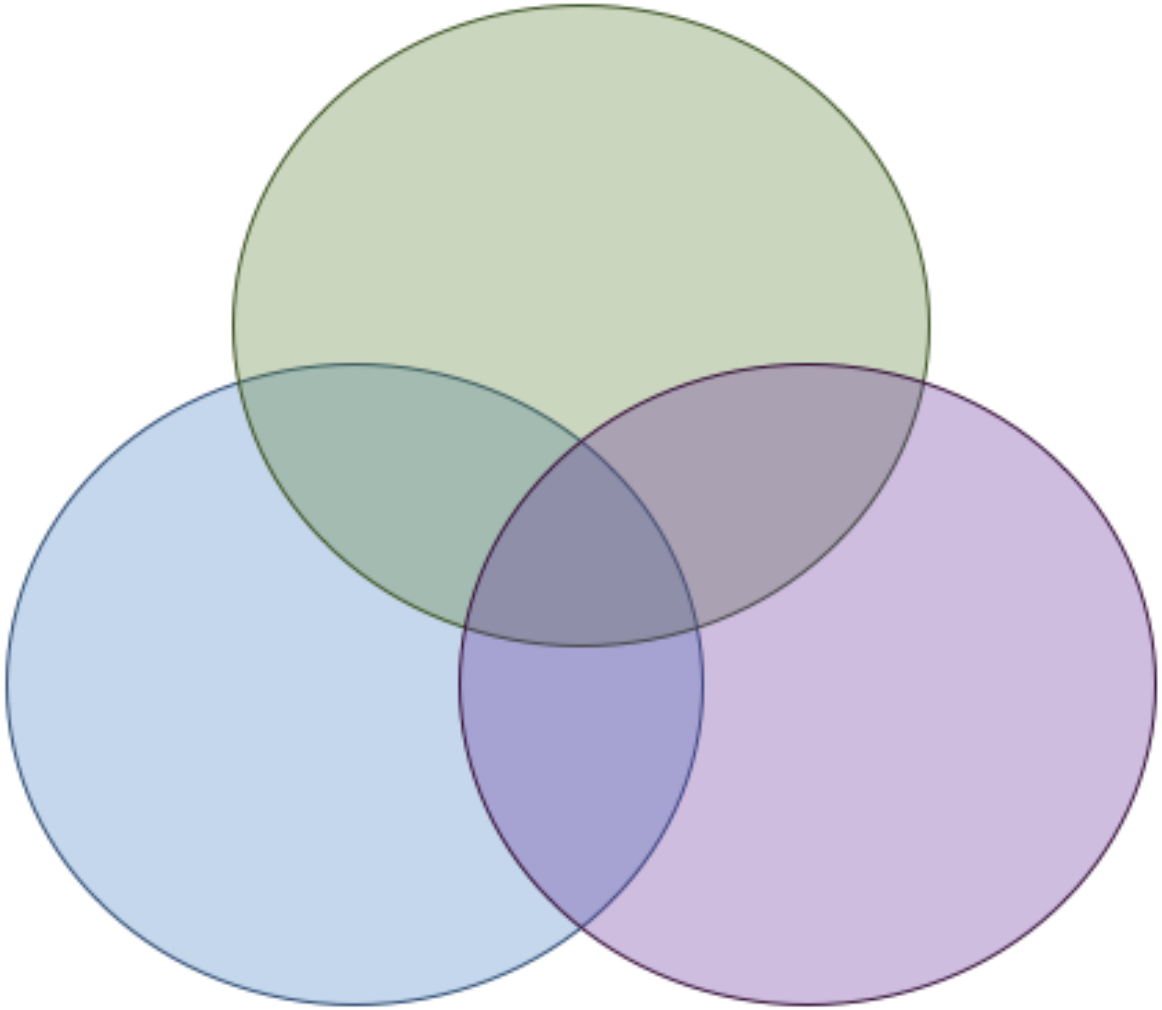
Quantum logarithmic Sobolev Inequalities for Quantum Many-Body Systems: An approach via Quasi-Factorization of the Relative Entropy

Ángela Capel Cuevas (ICMAT)

29 October 2019

Supervised by: David Pérez-García (UCM) and Angelo Lucia (Caltech)

Field of study



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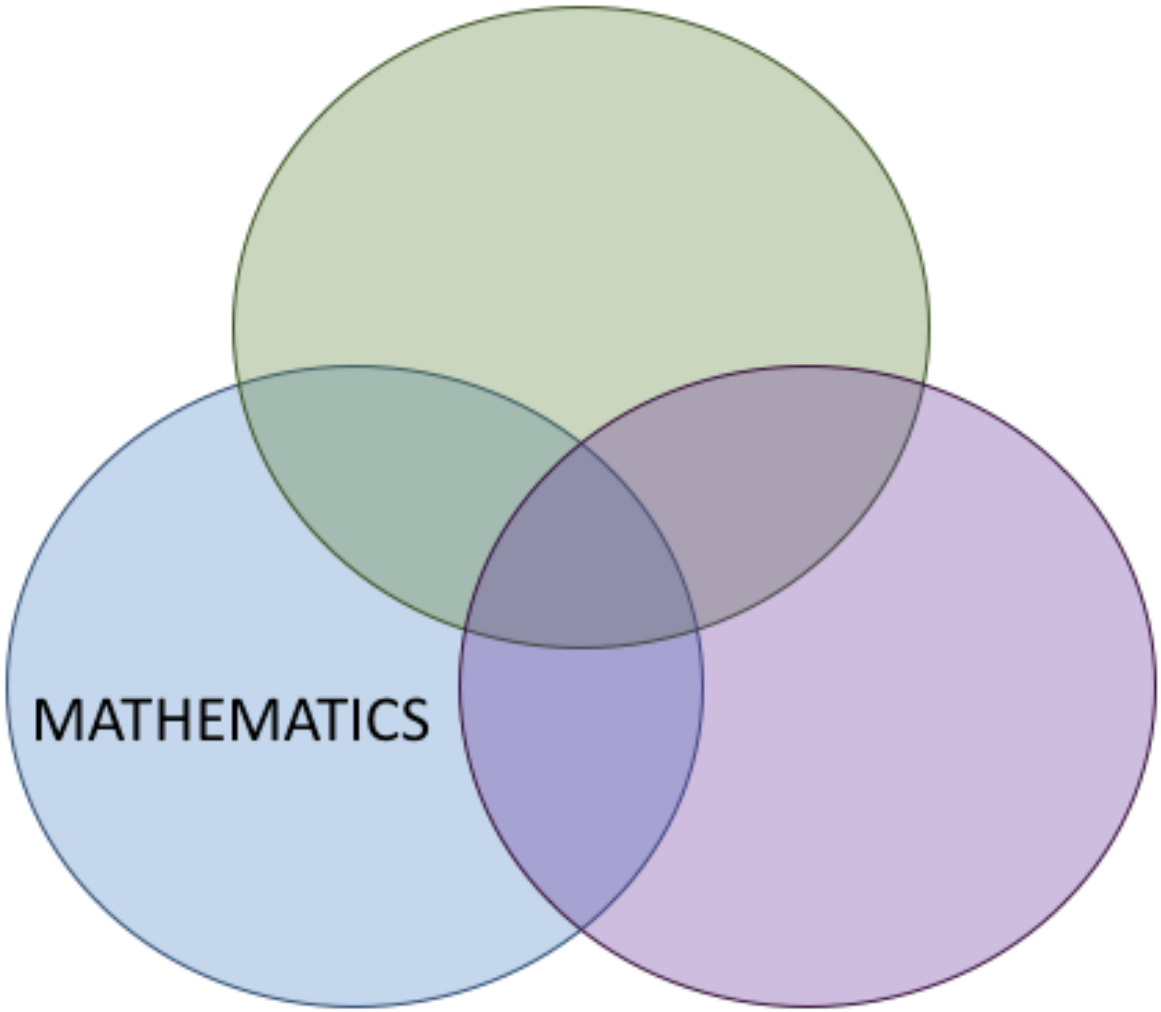

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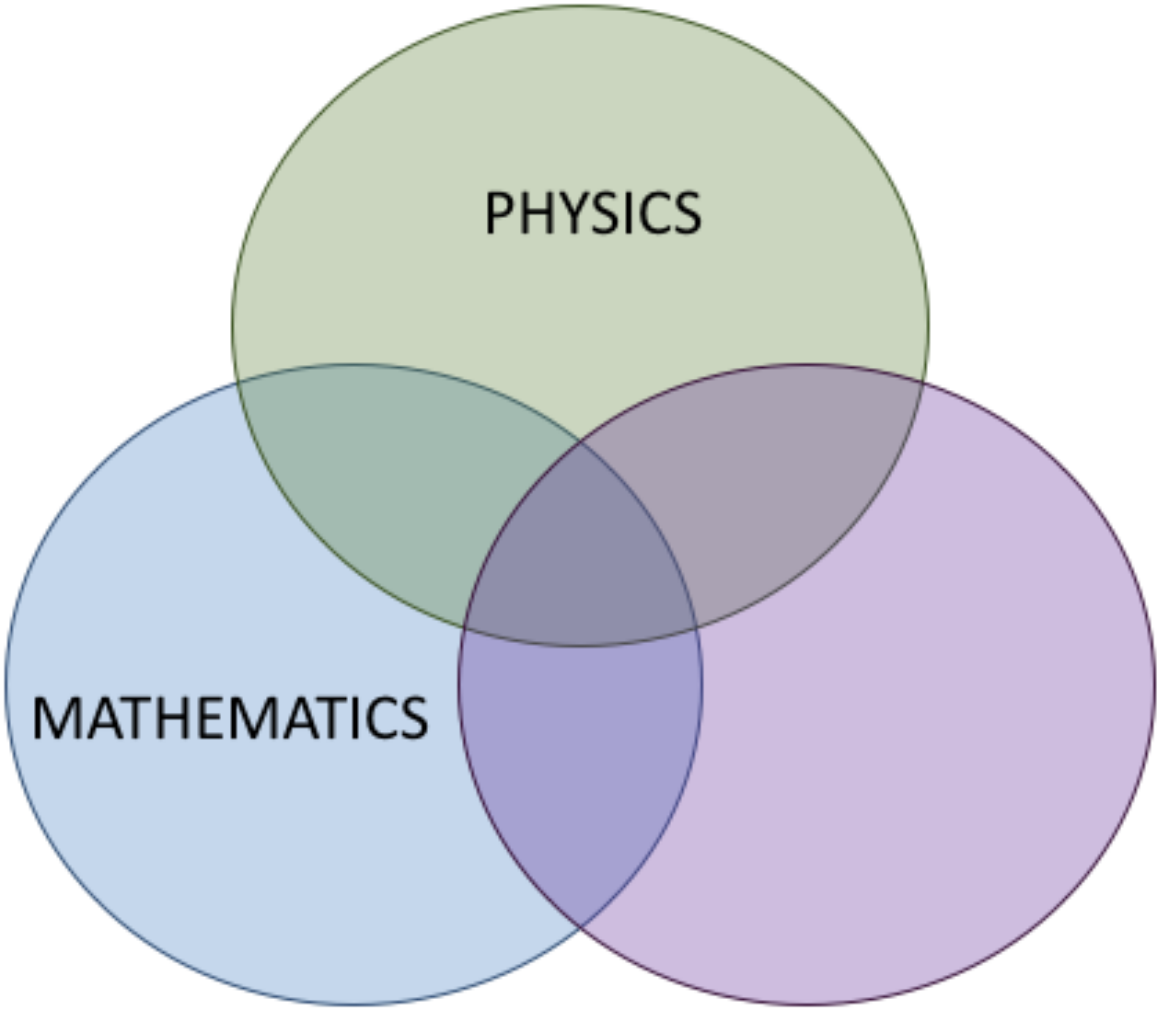

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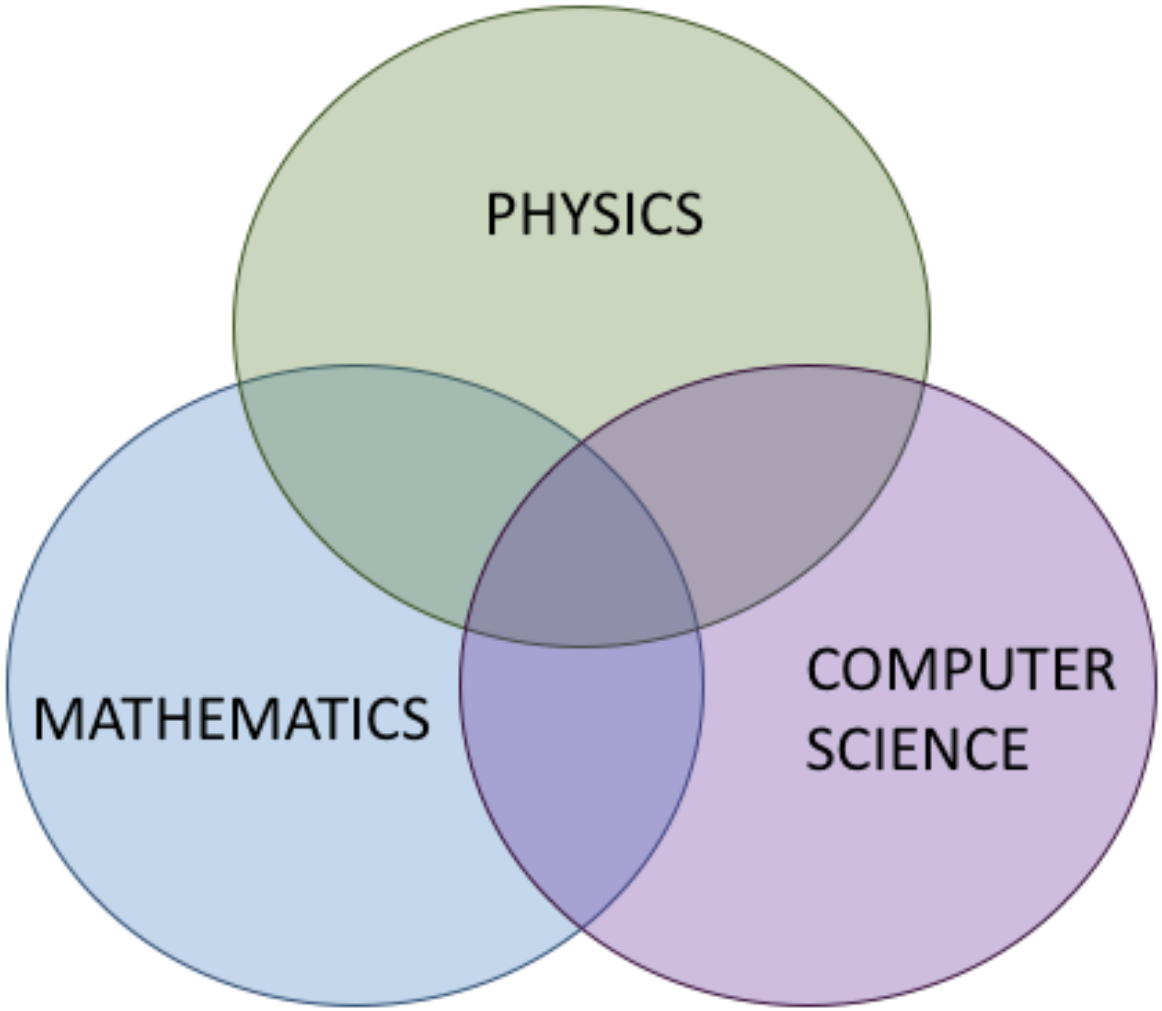

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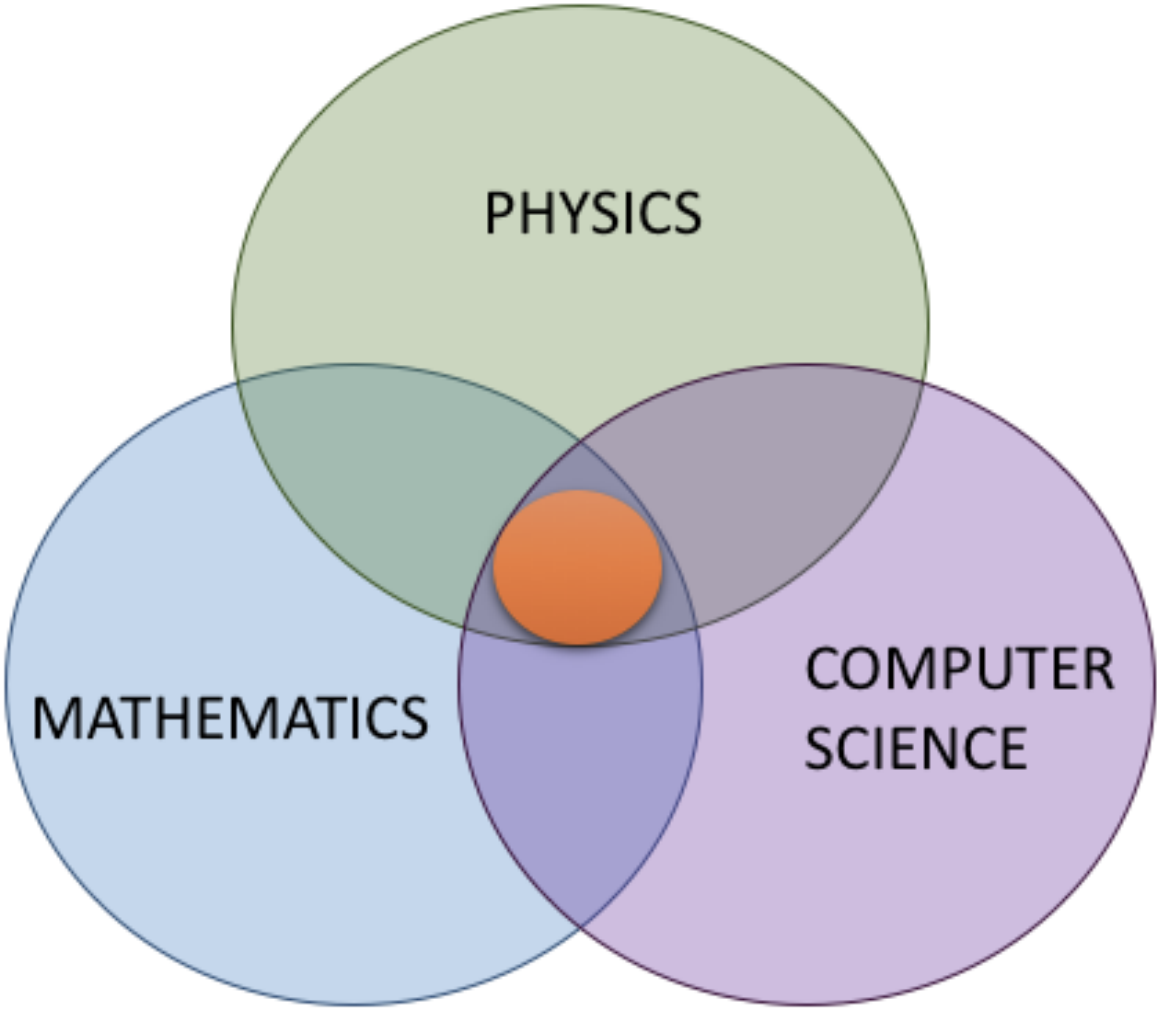

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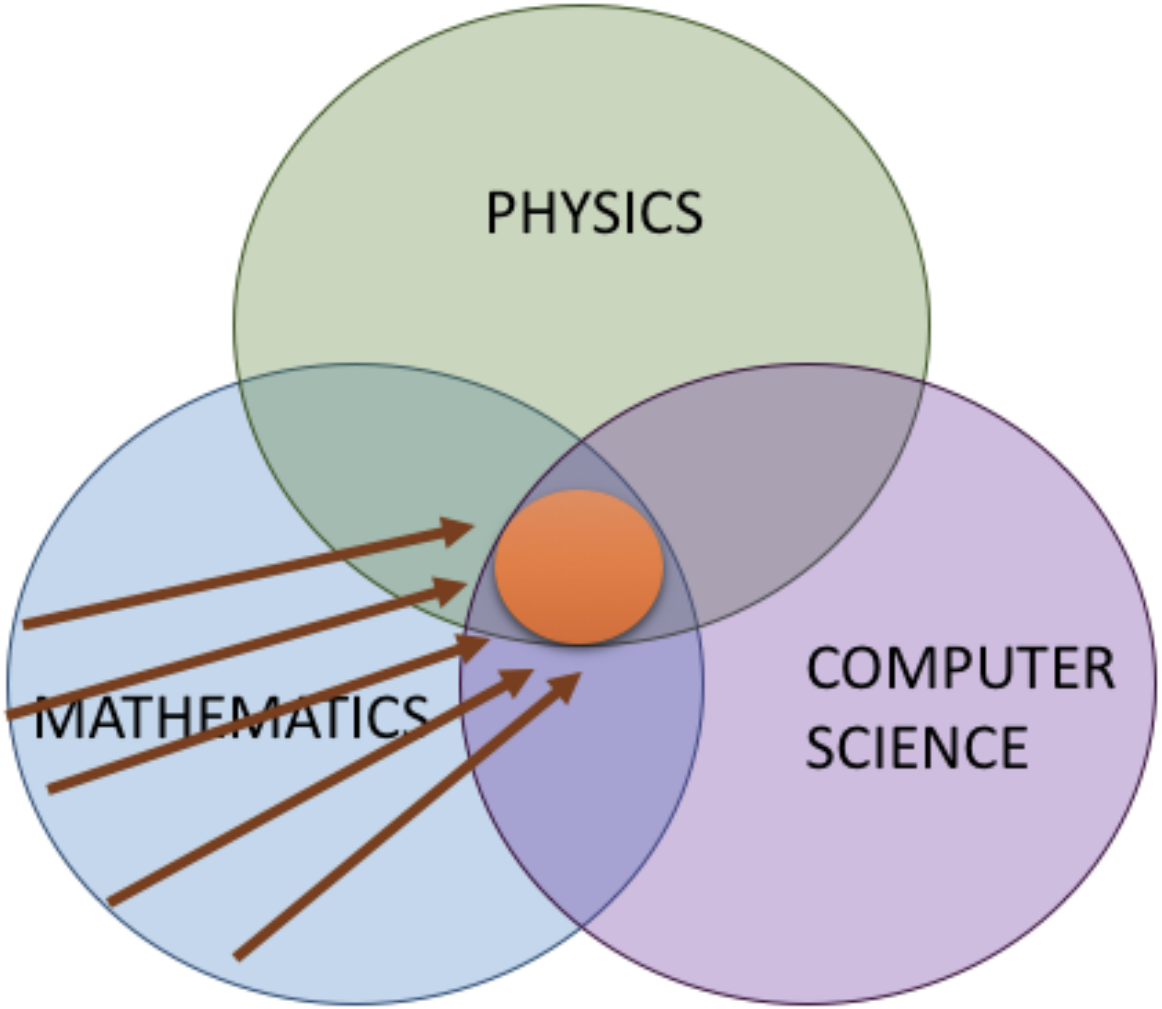

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Communication channels $\longleftrightarrow$ Physical interactions

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Dissipative evolutions of quantum many-body systems

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Velocity of convergence of certain quantum dissipative evolutions to their thermal equilibriums.

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Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant.

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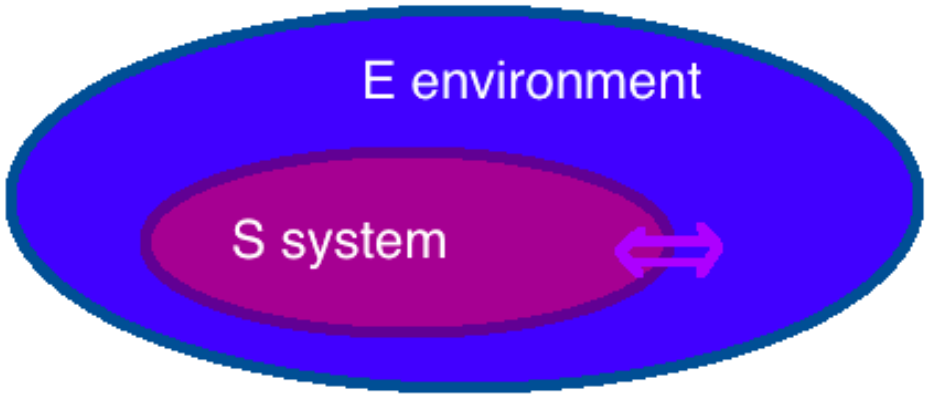
   - Quantum dissipative systems
   - Logarithmic Sobolev inequalities
2. Results
   - Strategy
   - Quasi-factorization of the relative entropy
   - Log-Sobolev constants
   - BS-entropy

1.1 Quantum dissipative systems

Open quantum systems

No experiment can be executed at zero temperature or be completely shielded from noise.

$\Rightarrow$ Open quantum many-body systems.



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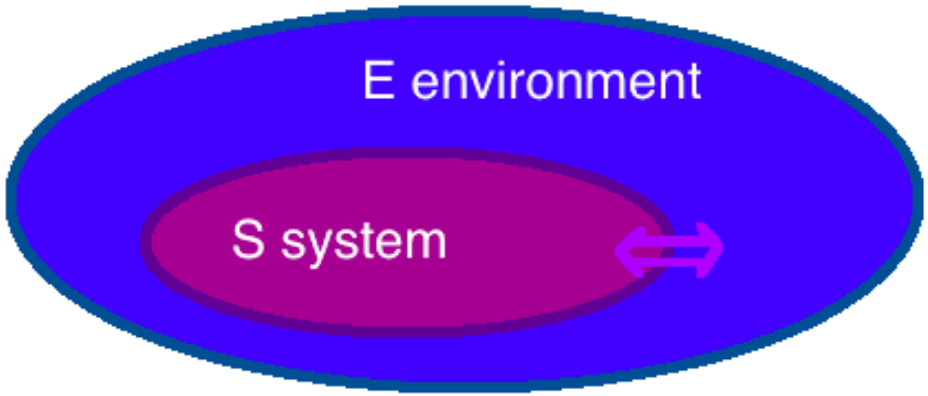

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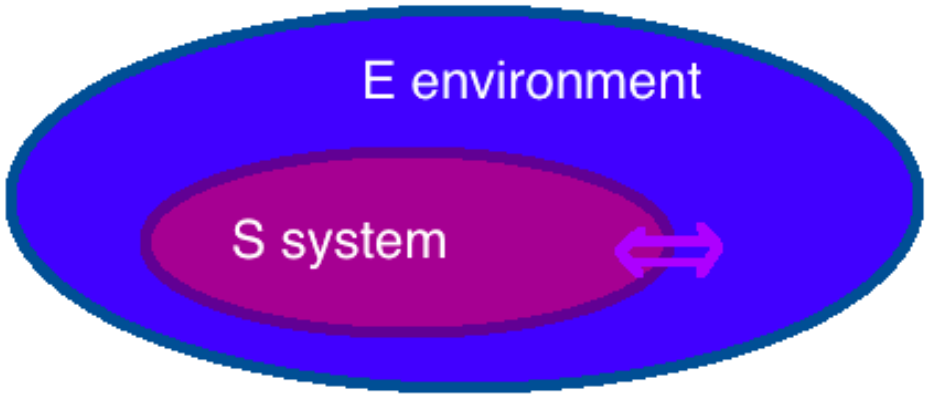

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Postulates of quantum mechanics

Postulate 1

Given an isolated physical system, there is a complex Hilbert space $\mathcal{H}$ associated to it, which is known as the **state space** of the system.

Moreover, the physical system is completely described by its **state vector**, which is a unitary vector in the state space.

Postulate 2

Given an isolated physical system, its evolution is described by a **unitary transformation** in the Hilbert space.

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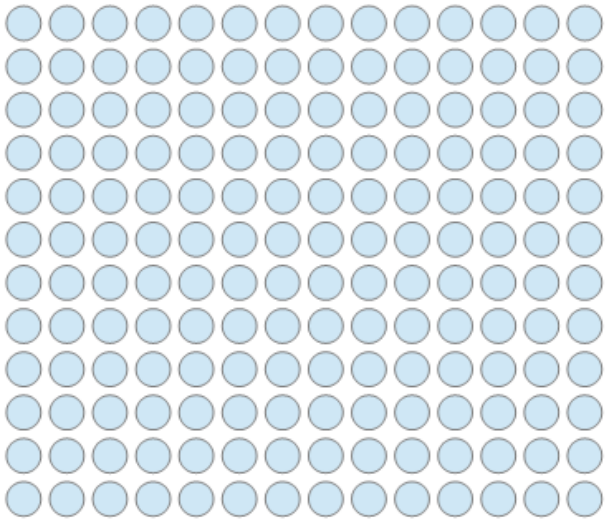

Figure: A quantum spin lattice system.

- Finite lattice $\Lambda \subset\subset \mathbb{Z}^d$.
- To every site $x \in \Lambda$ we associate $\mathcal{H}_x$ $(= \mathbb{C}^D)$.
- The global Hilbert space associated to Λ is $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$.
- The set of bounded linear endomorphisms on $\mathcal{H}_\Lambda$ is denoted by $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$.
- The set of density matrices is denoted by $\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{\rho_\Lambda \in \mathcal{B}_\Lambda \ : \ \rho_\Lambda \geq 0 \text{ and } \operatorname{tr}[\rho_\Lambda] = 1\}$.

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Physical evolution: $\rho \mapsto U \rho U^* \rightsquigarrow$ Reversible

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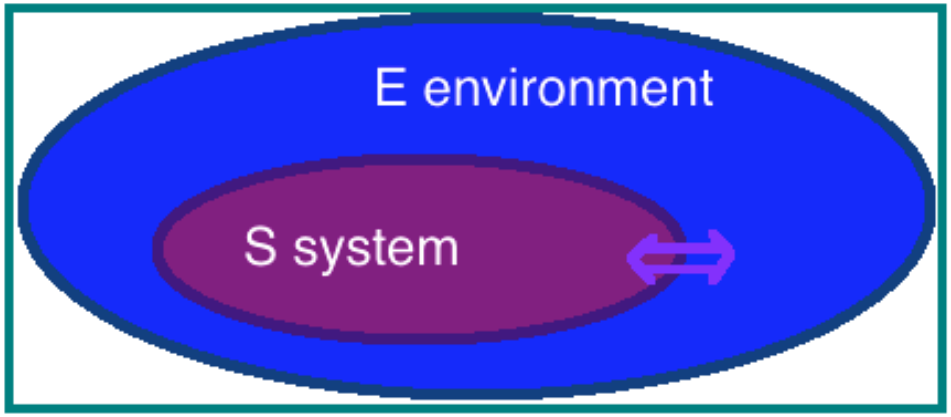
Open systems $\Rightarrow$ Environment and system interact.



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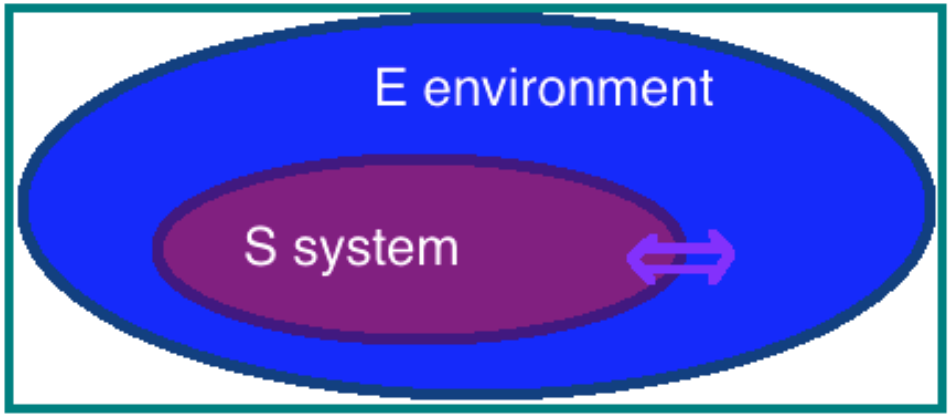

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State for the environment: $|\psi\rangle\langle\psi|_E$

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|_E \mapsto U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^* \mapsto \mathrm{tr}_E[U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^*] = \tilde{\rho}$$

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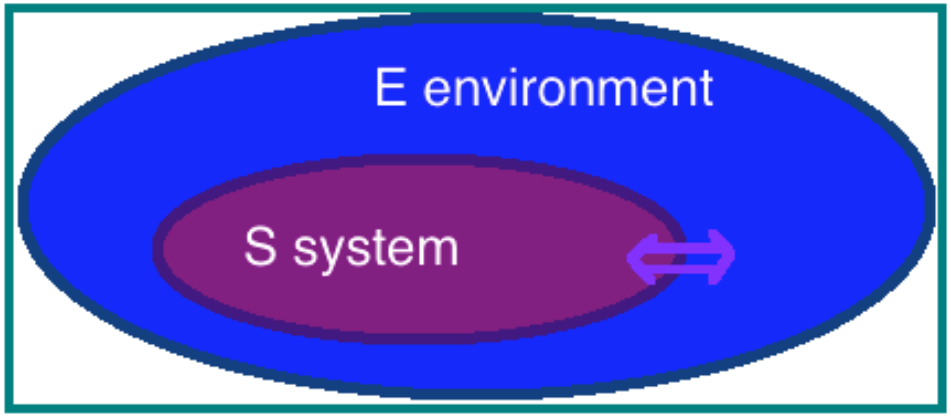

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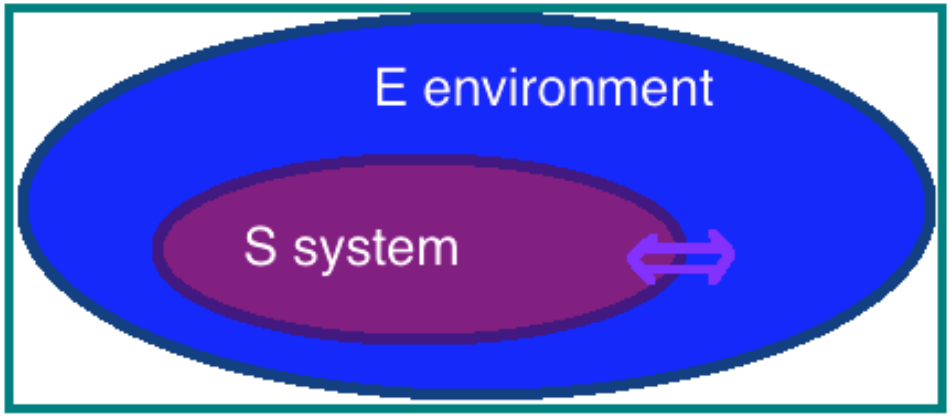

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Continuous-time description: For every $t \geq 0$, the corresponding time slice is a realizable evolution $\mathcal{T}_t$ (quantum channel).

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Primitive QMS

We assume that $\{\mathcal{T}_t^*\}_{t\geq 0}$ has a unique full-rank invariant state, which we denote by σ.

Reversibility

We also assume that the quantum Markov process studied is **reversible**, i.e., satisfies the **detailed balance condition**:

$$\langle f, \mathcal{L}(g)\rangle_\sigma = \langle \mathcal{L}(f), g\rangle_\sigma$$

for every $f, g \in \mathcal{A}$, in the Heisenberg picture.

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Quantum dissipative evolutions useful?

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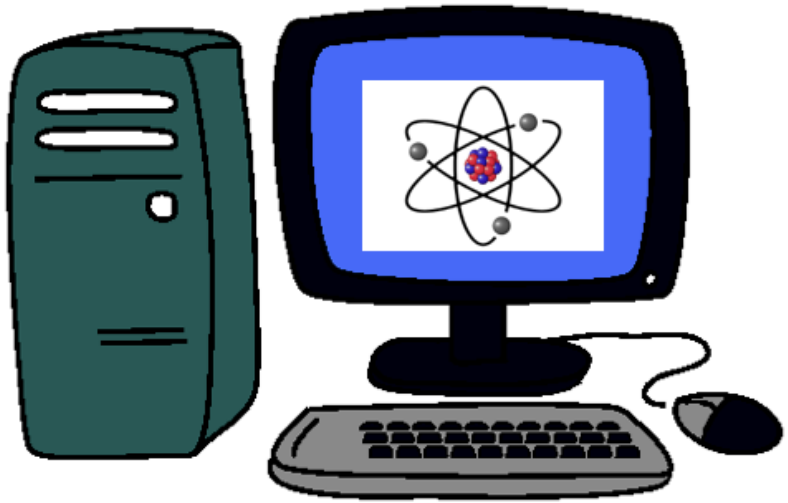

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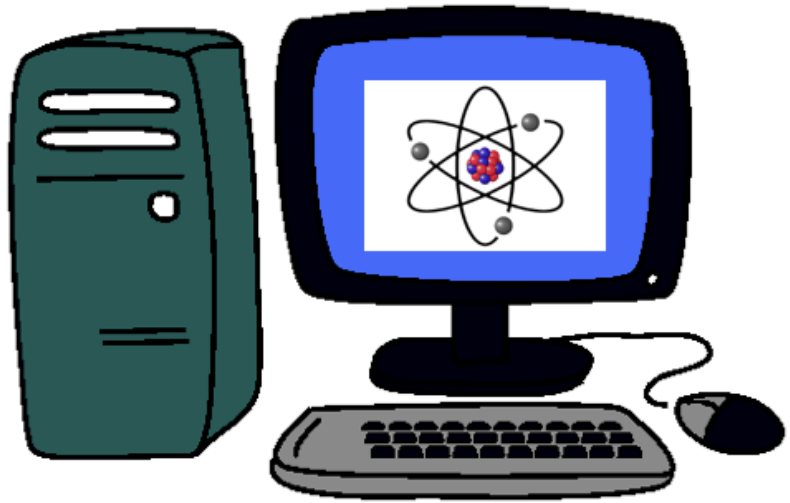

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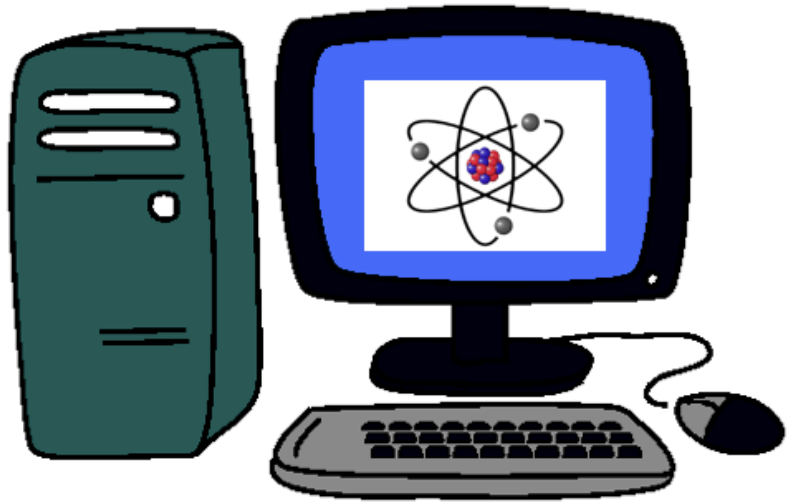

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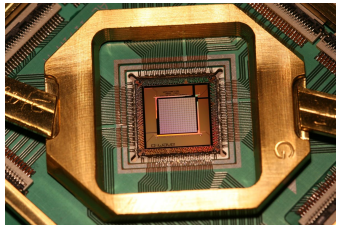

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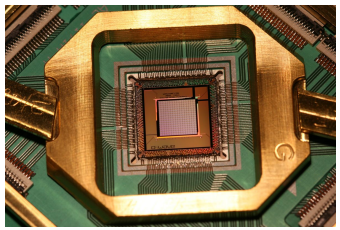

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- Conditions against noise
- Time to obtain certain states
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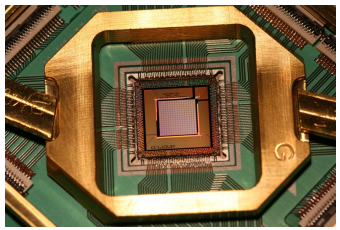

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Mixing time

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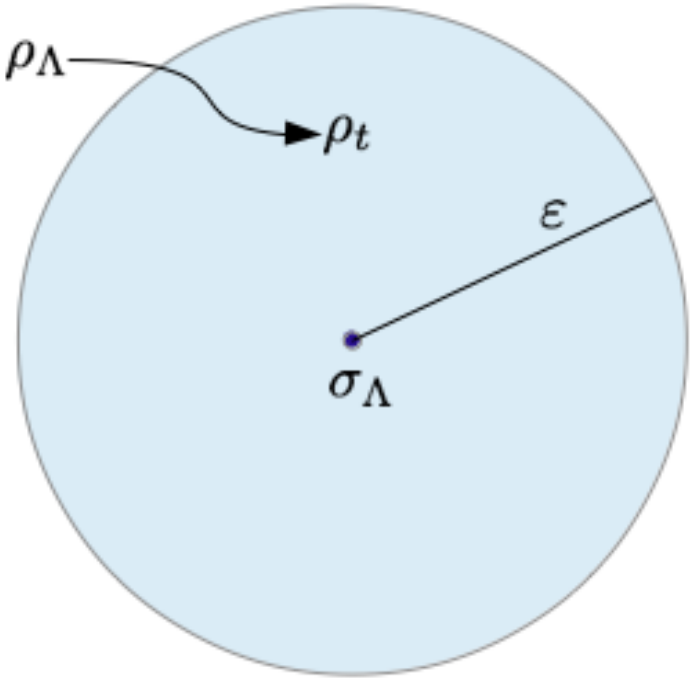
We define the **mixing time** of $\{\mathcal{T}_t^*\}$ by

$$\tau(\varepsilon) = \min\left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\mathcal{T}_t^*(\rho) - \mathcal{T}_\infty^*(\rho)\|_1 \leq \varepsilon \right\}.$$

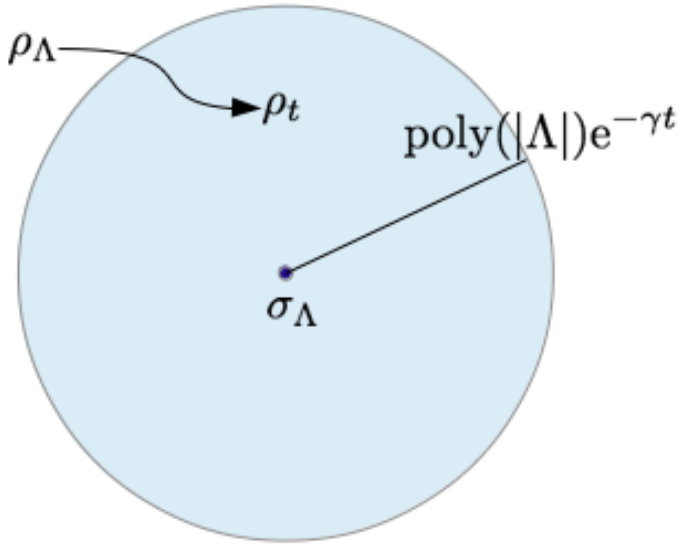


Rapid mixing

Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \text{poly}(|\Lambda|)e^{-\gamma t}.$$



Problem

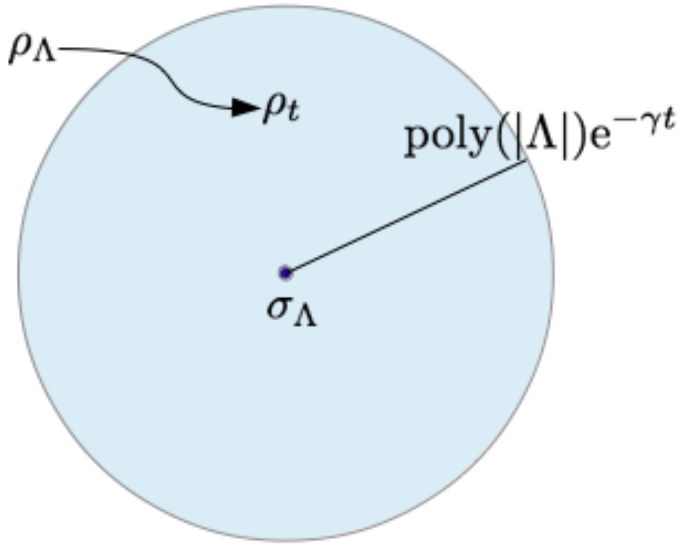
Find examples of rapid mixing!

Rapid mixing

Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\rho_t - \sigma_\Lambda\|_1 \leq \text{poly}(|\Lambda|)e^{-\gamma t}.$$



Problem

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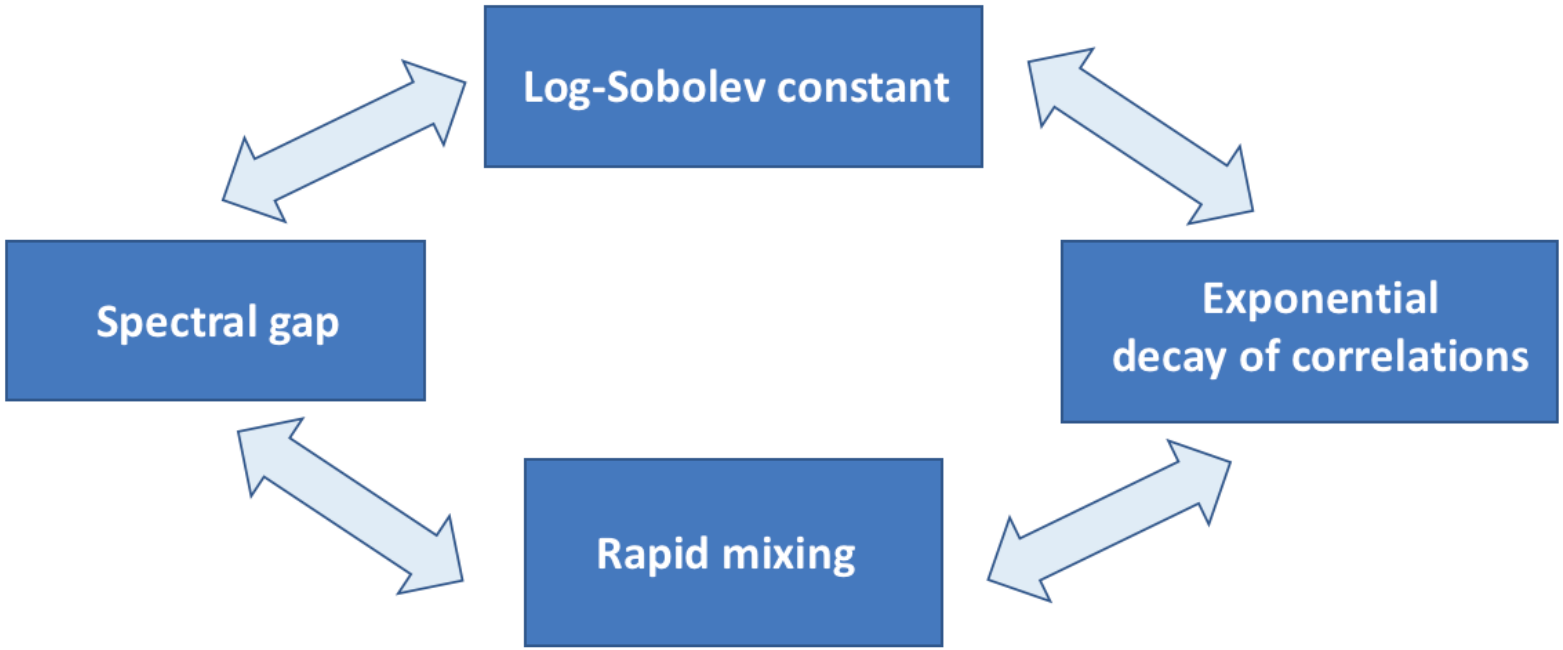
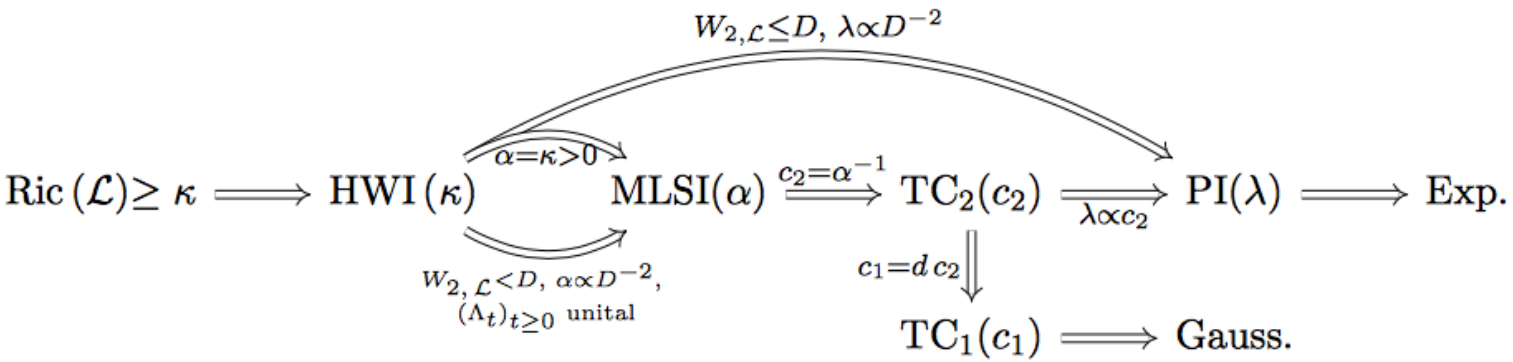
1.2 Logarithmic Sobolev inequalities

Classical spin systems



Quantum spin systems



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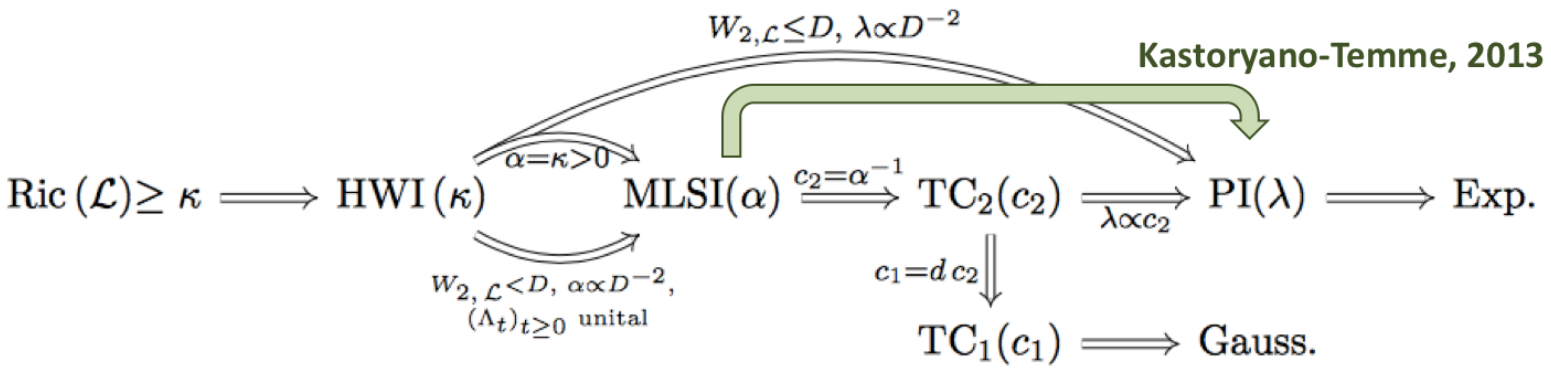

Log-Sobolev inequality (MLSI)

Recall: $\rho_t := \mathcal{T}_t^*(\rho)$.

Liouville's equation:

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Lower bound for the derivative of $D(\rho_t||\sigma_\Lambda)$ in terms of itself:

$$2\alpha D(\rho_t||\sigma_\Lambda) \leq -\mathrm{tr}[\mathcal{L}_\Lambda^*(\rho_t)(\log \rho_t - \log \sigma_\Lambda)]. \quad (2)$$

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Log-Sobolev constant

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If $\alpha(\mathcal{L}_\Lambda^*) > 0$:

$$D(\rho_t||\sigma_\Lambda) \leq D(\rho_\Lambda||\sigma_\Lambda) e^{-2\,\alpha(\mathcal{L}_\Lambda^*)\,t},$$

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and with **Pinsker's inequality**, we have:

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2D(\rho_\Lambda||\sigma_\Lambda)}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t} \leq \sqrt{2\log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t}.$$

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First main objective of this thesis

Develop a strategy to find positive log Sobolev constants from static properties on the fixed point.

Second main objective of this thesis

Apply that strategy to certain dissipative dynamics.

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2 Results

Based on:

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2.1 Strategy

Classical spin systems

(Cesi, Dai Pra-Paganoni-Posta, '02)

(1) Quasi-factorization of the entropy (in terms of a conditional entropy).

+

(2) Recursive geometric argument.
Lower bound for the global log-Sobolev constant in terms of the log-Sobolev constant of a size-fixed region.

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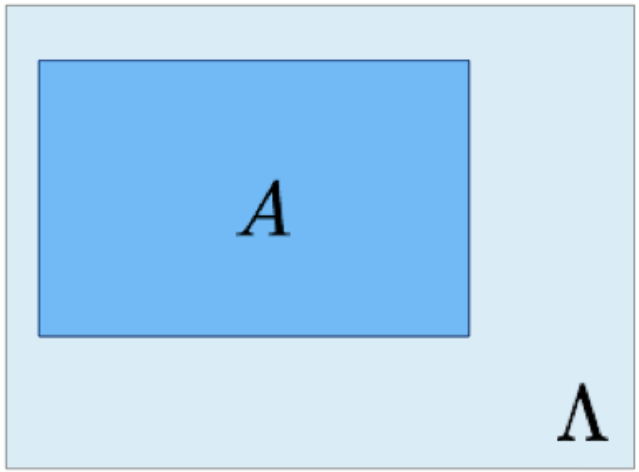
Positive log-Sobolev constant.

Objective

What do we want to prove?

$$\liminf_{\Lambda \nearrow \mathbb{Z}^d} \alpha(\mathcal{L}_\Lambda^*) \geq \Psi(|\Lambda|) > 0.$$



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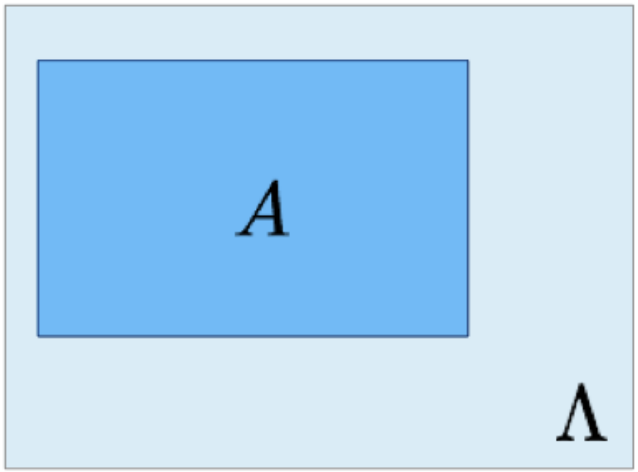

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Conditional log-Sobolev constant

Log-Sobolev constant

Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ. We define the **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ by

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$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda||\sigma_\Lambda)}$$

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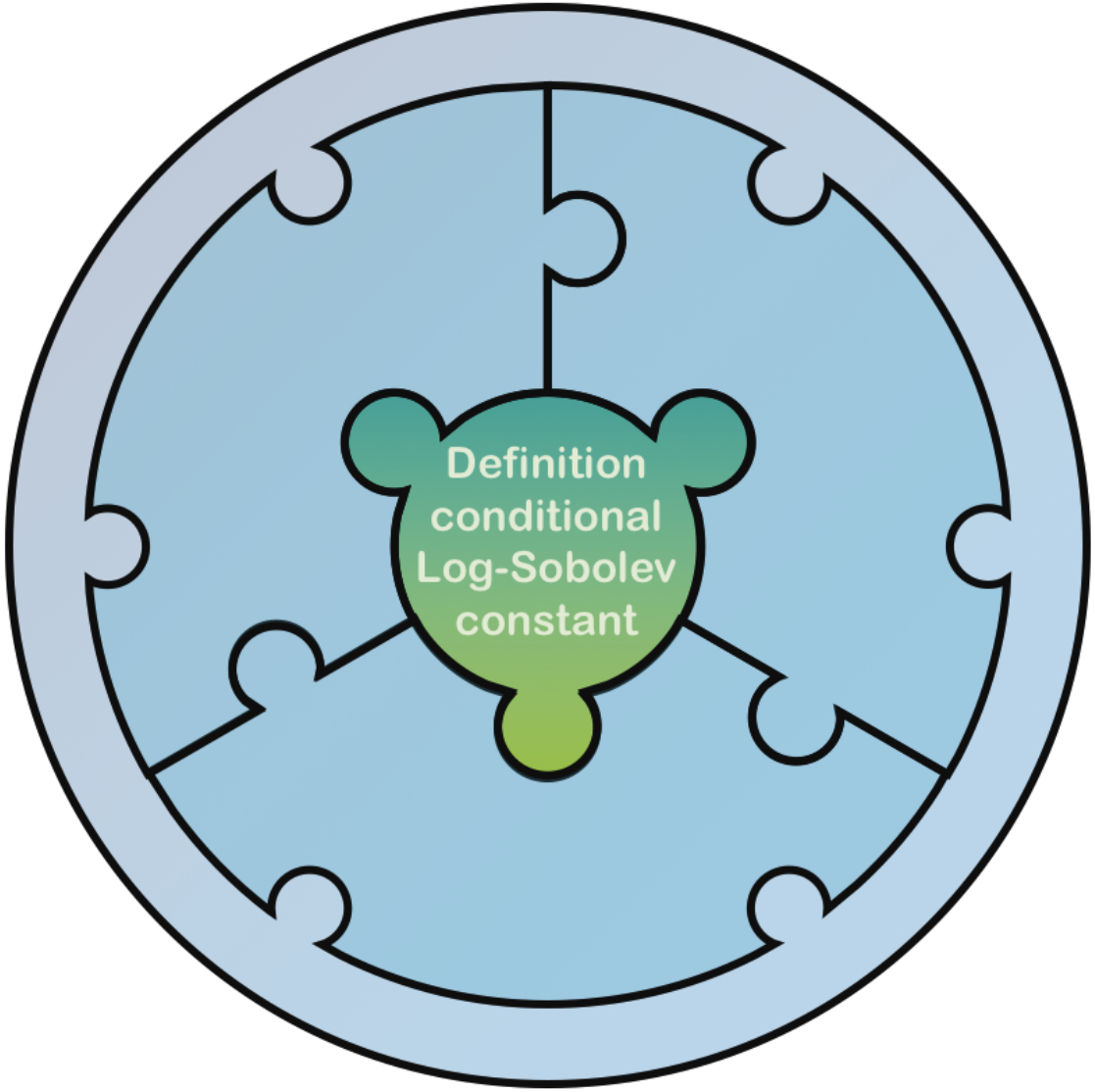

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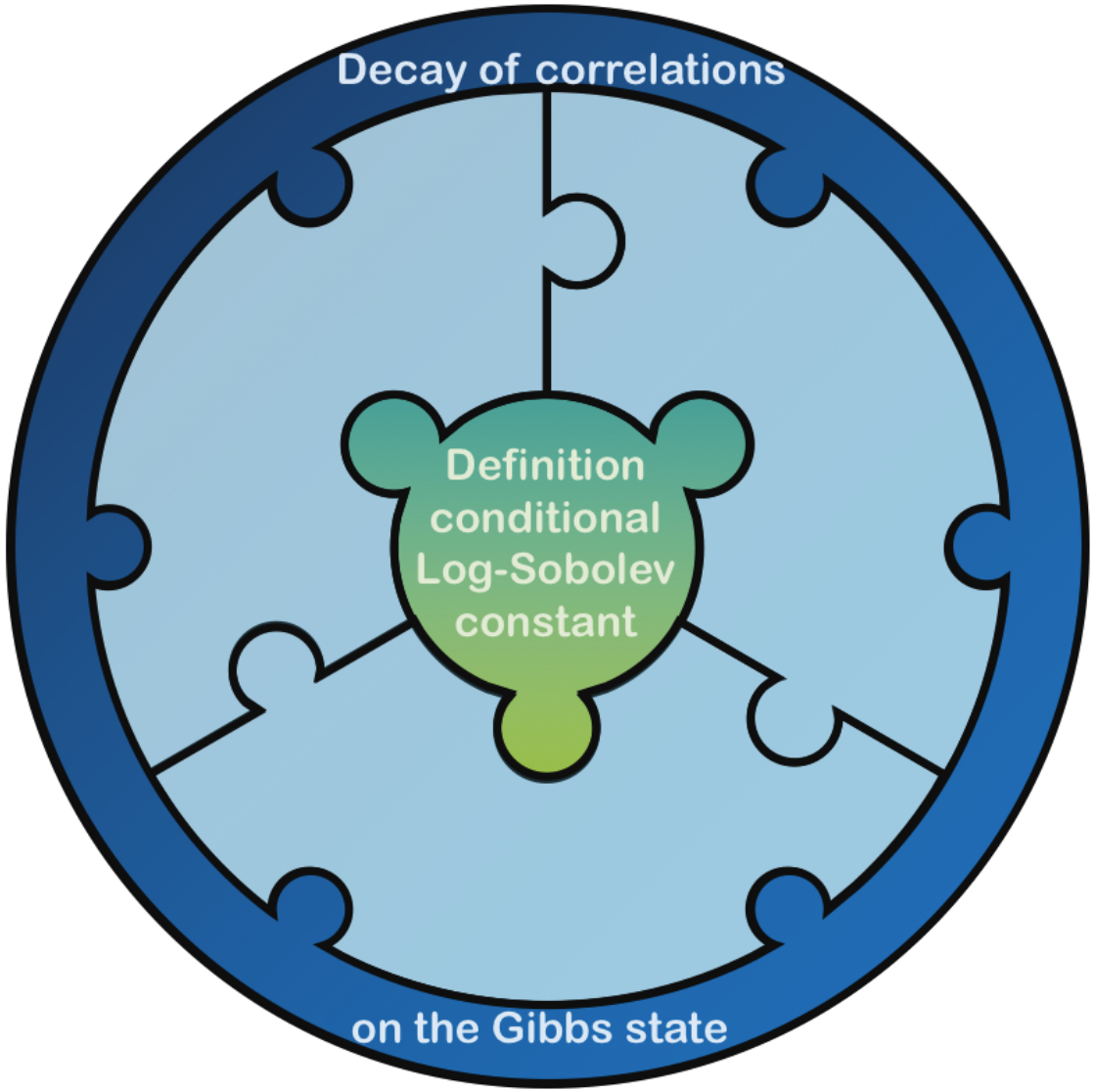
Decay of correlations
Definition conditional Log-Sobolev constant
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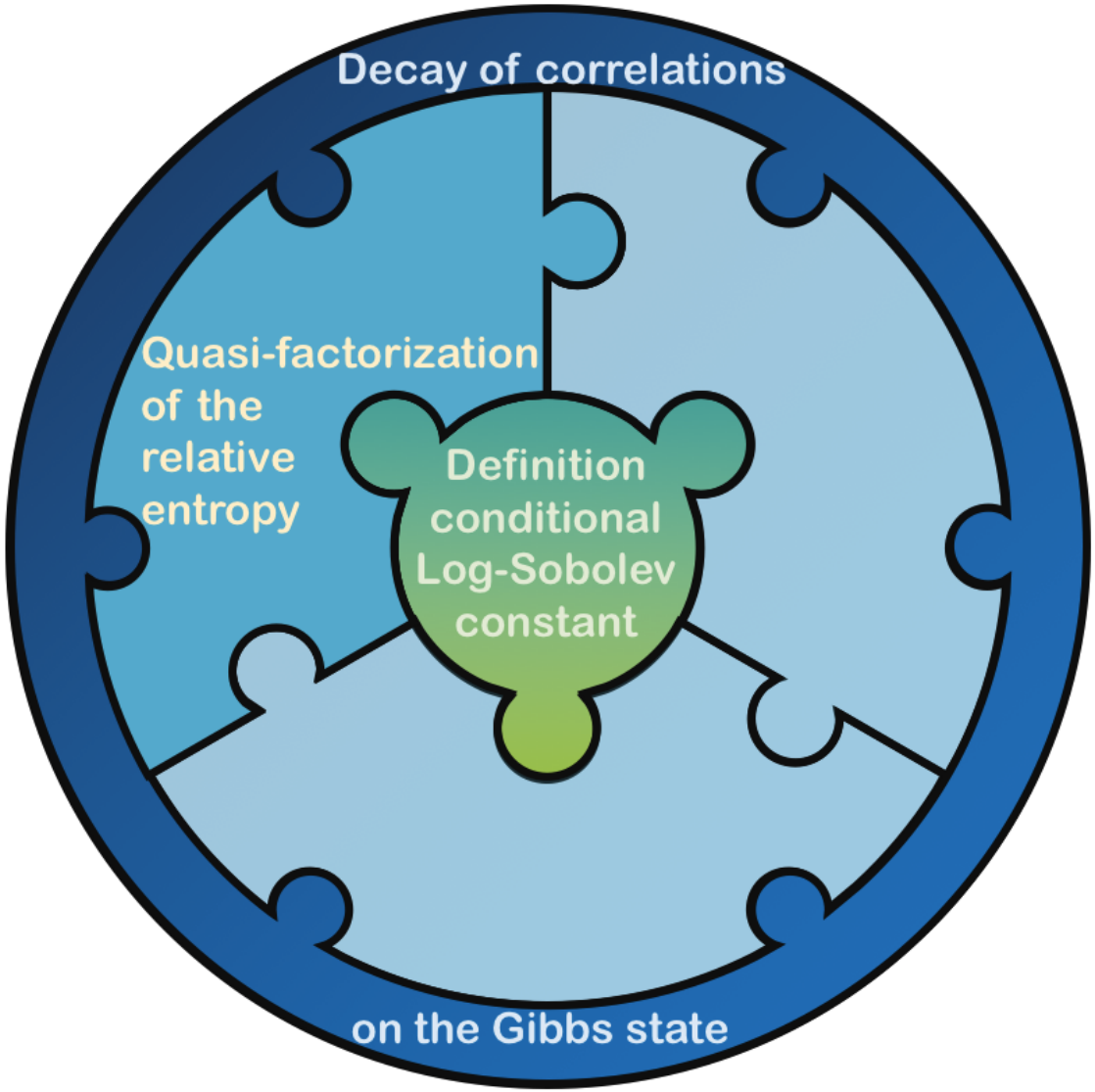
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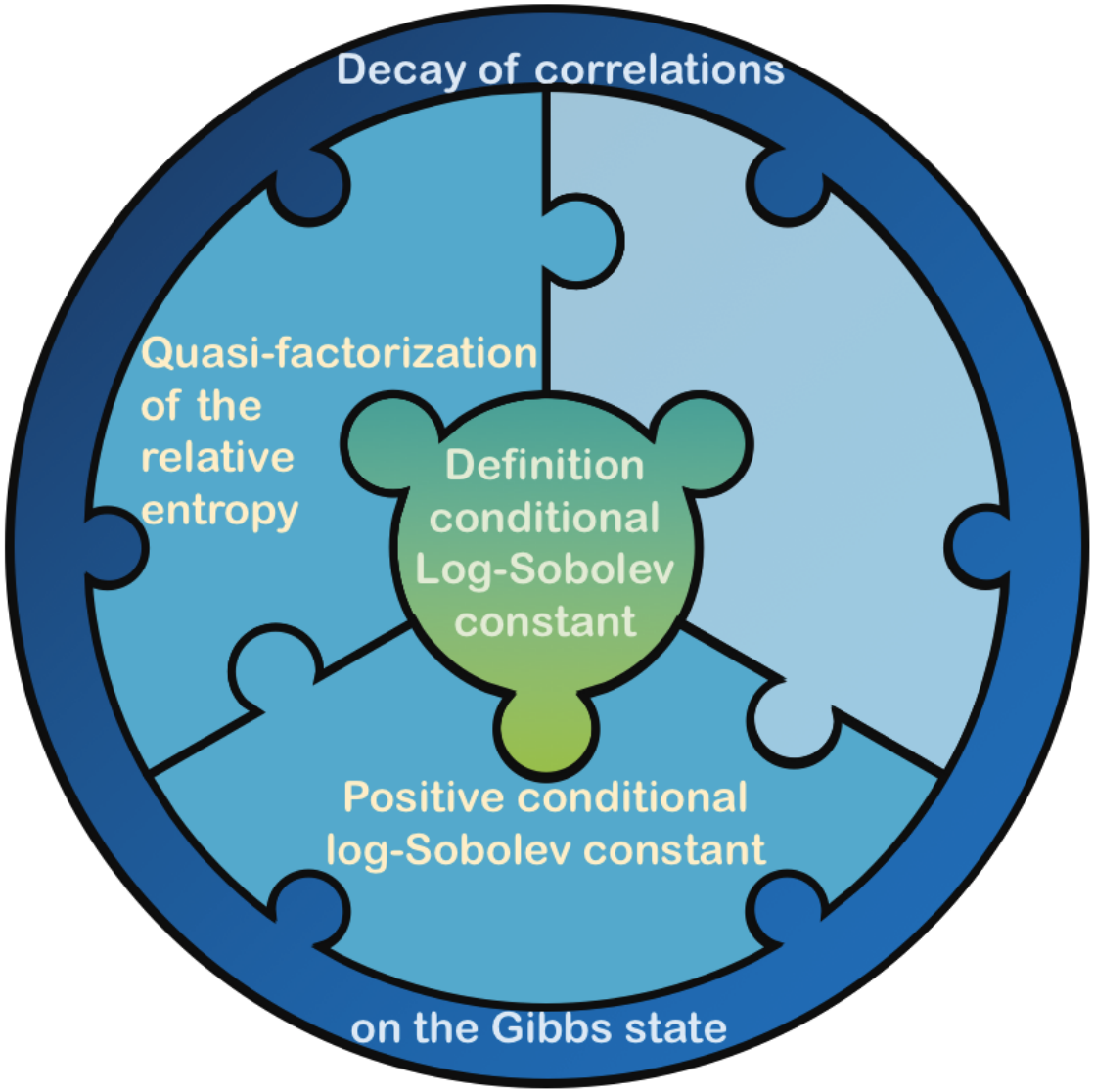
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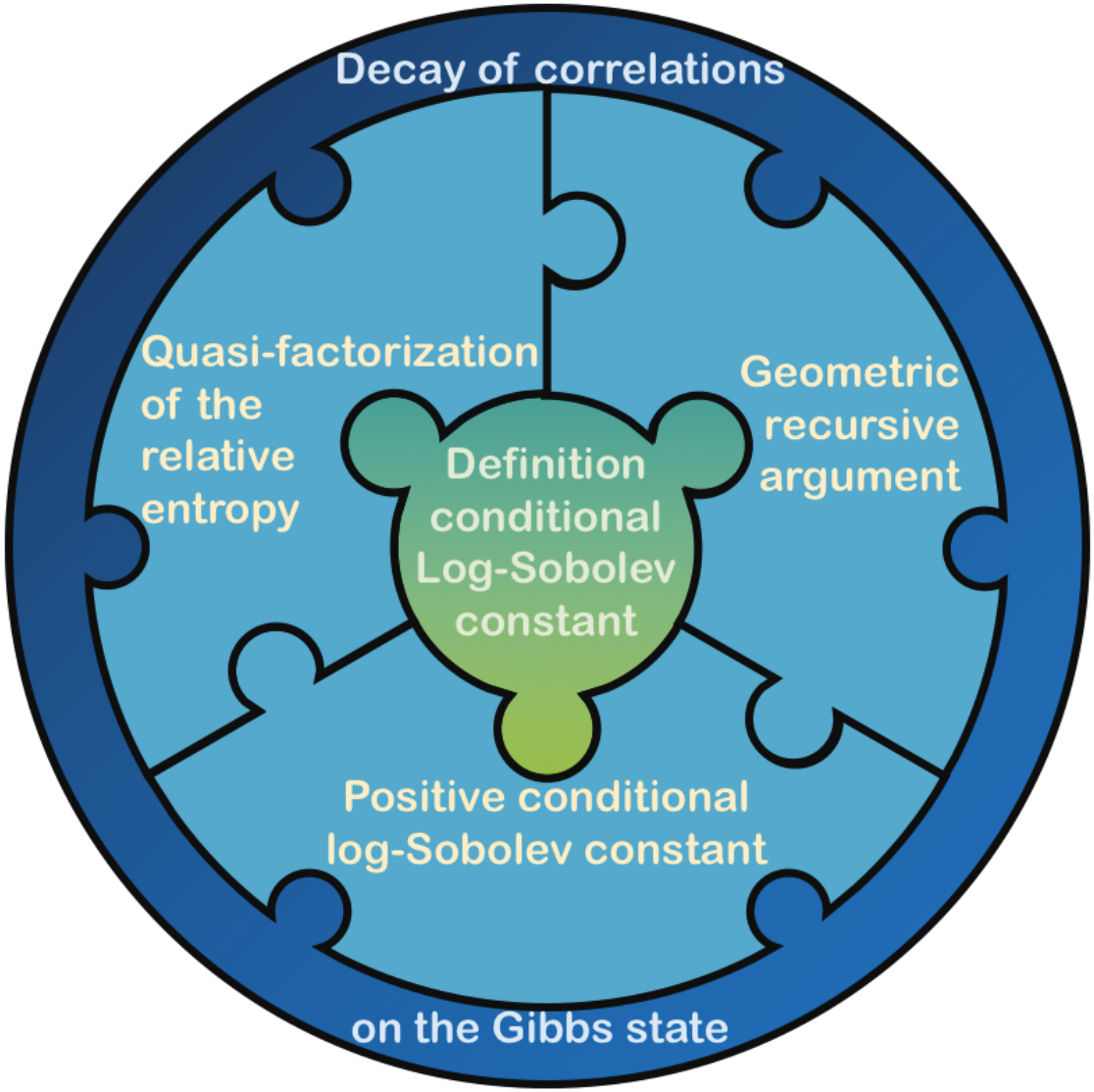
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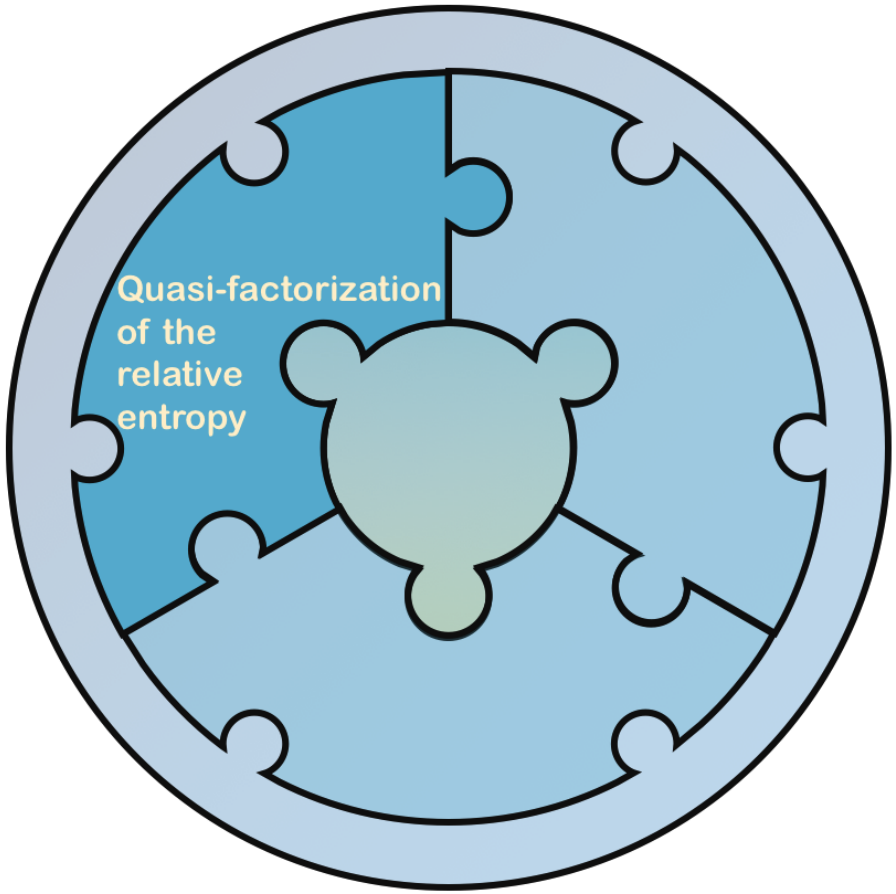
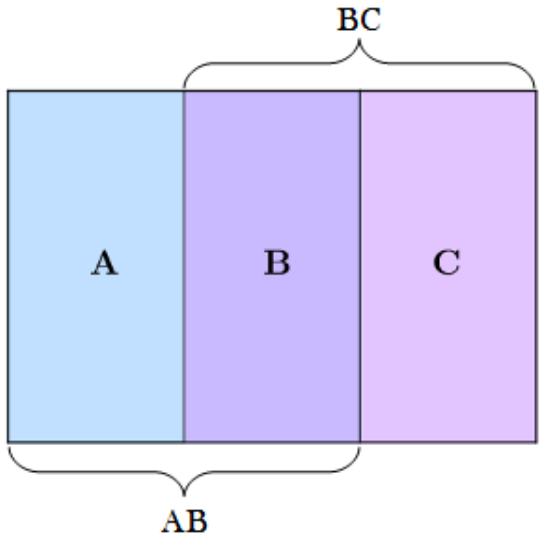
2.2 Part 2: Quasi-factorization of the relative entropy



Statement of the problem



Problem

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in S_{ABC}$. Can we prove something like

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC})\right] \ ?$$

Quantum relative entropy

$$D(\rho||\sigma) = \mathrm{tr}\left[\rho(\log\rho - \log\sigma)\right]$$

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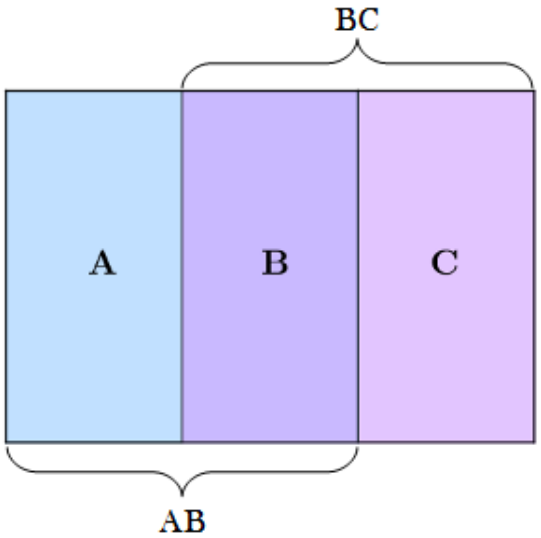

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Classical case, Dai Pra et al. '02

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{1 - 4\|h-1\|_\infty}\, \mu\left[\mathrm{Ent}_\mu(f \mid \mathcal{F}_1) + \mathrm{Ent}_\mu(f \mid \mathcal{F}_2)\right],$$

where $h = \dfrac{d\mu}{d\bar{\mu}}$.

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Classical entropy and conditional entropy

Entropy:

$$\mathrm{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Conditional entropy:

$$\mathrm{Ent}_\mu(f \mid \mathcal{G}) = \mu(f \log f \mid \mathcal{G}) - \mu(f \mid \mathcal{G}) \log \mu(f \mid \mathcal{G}).$$

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Relative entropy

Quantum relative entropy

Let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$. The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

$$D(\rho_\Lambda||\sigma_\Lambda) = \mathrm{tr}\left[\rho_\Lambda(\log \rho_\Lambda - \log \sigma_\Lambda)\right].$$

Properties of the relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** $\rho_{AB} \mapsto D(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity.** $D(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
4. **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB}))$ for every quantum channel $\mathcal{T}$.

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$$D(\rho_\Lambda||\sigma_\Lambda) = \mathrm{tr}\left[\rho_\Lambda(\log\rho_\Lambda - \log\sigma_\Lambda)\right].$$

Properties of the relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** $\rho_{AB} \mapsto D(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity.** $D(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
4. **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB}))$ for every quantum channel $\mathcal{T}$.

Characterization of the RE, Wilming et al. '17, Matsumoto '10

If $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$ satisfies $1-4$, then f is the relative entropy.

Relative entropy

Quantum relative entropy

Let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$. The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

$$D(\rho_\Lambda||\sigma_\Lambda) = \mathrm{tr}\left[\rho_\Lambda(\log\rho_\Lambda - \log\sigma_\Lambda)\right].$$

Properties of the relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** $\rho_{AB} \mapsto D(\rho_{AB}||\sigma_{AB})$ is continuous.
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Characterization of the RE, Wilming et al. '17, Matsumoto '10

If $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$ satisfies $1-4$, then f is the relative entropy.

Conditional relative entropy

Conditional relative entropy, (Q-Fact)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a **conditional relative entropy** in A as a function

$$D_A(\cdot||\cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and
   (2.1) $D_A(\rho_{AB}||\sigma_{AB})=0$ if, and only if, $\rho_{AB} = \sigma_{AB}^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma_{AB}^{1/2}$.
3. **Semi-superadditivity:** $D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$ and
   (3.1) **Semi-additivity:** if $\rho_{AB} = \rho_A \otimes \rho_B$, $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.
4. **Semi-motonicity:** For every quantum channel $\mathcal{T}$,

$$D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) + D_B((\mathrm{tr}_A \circ \mathcal{T})(\rho_{AB})||(\mathrm{tr}_A \circ \mathcal{T})(\sigma_{AB})) \leq D_A(\rho_{AB}||\sigma_{AB}) + D_B(\mathrm{tr}_A(\rho_{AB})||\,\mathrm{tr}_A(\sigma_{AB})).$$

Remark

Consider for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$

$$D^{+}_{A,B}(\rho_{AB}||\sigma_{AB}) = D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}).$$

Then, $D^{+}_{A,B}$ verifies the following properties:

1. **Continuity:** $\rho_{AB} \mapsto D^{+}_{A,B}(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity:** $D^{+}_{A,B}(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity:** $D^{+}_{A,B}(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

However, it does not satisfy the property of monotonicity.

Axiomatic characterization of the CRE, (Q-Fact)

The only possible conditional relative entropy is given by:

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B)$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

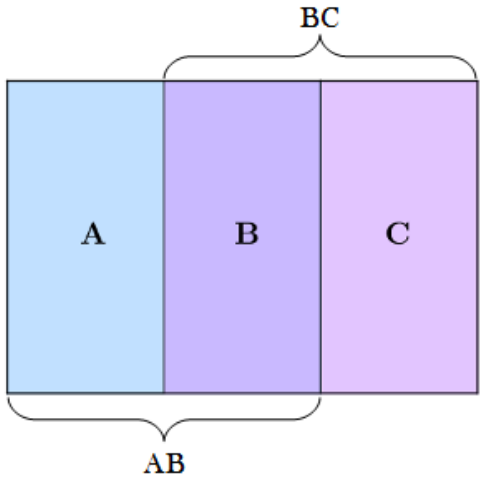

Figure: Choice of indices in $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Result of **quasi-factorization** of the relative entropy, for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$:

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC})\right],$$

where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

Quasi-factorization for the CRE, (Q-Fact)

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \frac{1}{1 - 2\|H(\sigma_{AC})\|_\infty} \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC})\right],$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2}\, \sigma_{AC}\, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Note that $H(\sigma_{AC}) = 0$ if σ_{AC} is a tensor product between A and C.

$$(1-2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \leq$$
$$D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) =$$
$$= 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).$$

$$\Leftrightarrow$$

$$(1+2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

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$$\Leftrightarrow$$

$$(1+2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

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$$(1-2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \leq$$
$$D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) =$$
$$= 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).$$

$$\Leftrightarrow$$

$$(1+2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

$$\Leftrightarrow$$

$$(1+2\|H(\sigma_{AC})\|_\infty)D(\rho_{AC}||\sigma_{AC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

This result is equivalent to **(Super)**:

$$\boxed{(1+2\|H(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

This result is equivalent to **(Super)**:

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Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

Due to:

- **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

we have

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

This result is equivalent to **(Super)**:

$$\boxed{(1+2\|H(\sigma_{AB})\|_{\infty})D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

Due to:

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we have

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

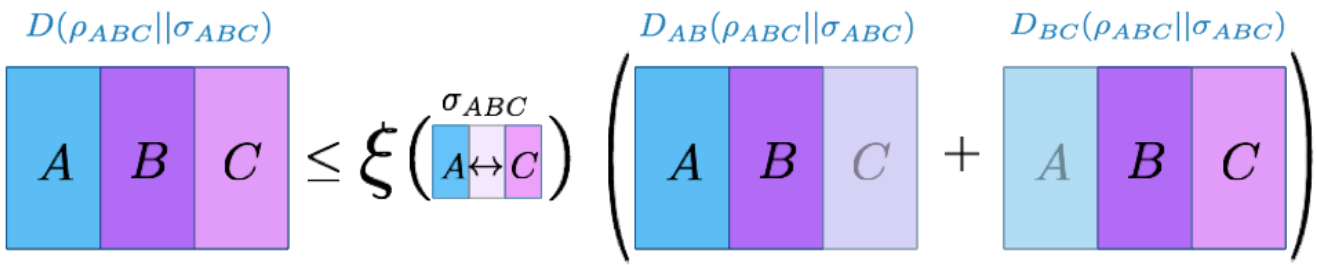
Quasi-factorization for the CRE (Q-Fact)

Let $\mathcal{H}_{ABC}$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. The following holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{AC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC})\right],$$

where

$$\xi(\sigma_{AC}) = \frac{1}{1 - 2\left\|\sigma_A^{-1/2} \otimes \sigma_C^{-1/2}\, \sigma_{AC}\, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}\right\|_\infty}.$$



Weak conditional relative entropy

Weak conditional relative entropy, (Q-Fact)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a **conditional relative entropy** in A as a function

$$D_A(\cdot||\cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and
   (2.1) $D_A(\rho_{AB}||\sigma_{AB})=0$ if, and only if, $\rho_{AB} = \sigma_{AB}^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma_{AB}^{1/2}$.
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Conditional relative entropy by expectations

Conditional relative entropy by expectations, (Q-Fact)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in S_{AB}$. Let $\mathbb{E}_A^*$ be defined as

$$\mathbb{E}_A^*(\rho_{AB}) := \sigma_{AB}^{1/2} \sigma_B^{-1/2} \rho_B \sigma_B^{-1/2} \sigma_{AB}^{1/2}. \tag{3}$$

We define the **conditional relative entropy by expectations** of ρ_{AB} and σ_{AB} in A by:

$$D_A^E(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\mathbb{E}_A^*(\rho_{AB})).$$

Property

$D_A^E(\rho_{AB}||\sigma_{AB})$ is a weak conditional relative entropy.

Quasi-factorization CRE by expectations, (Q-Fact)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following inequality holds

$$(1 - \xi(\sigma_{AB}))D(\rho_{AB}||\sigma_{AB}) \leq D_A^E(\rho_{AB}||\sigma_{AB}) + D_B^E(\rho_{AB}||\sigma_{AB}), \tag{4}$$

where

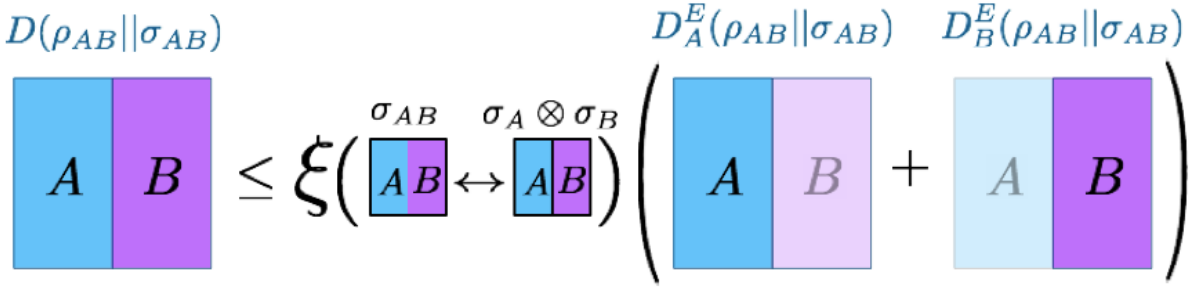
$$\xi(\sigma_{ABC}) = 2\,(E_1(t) + E_2(t)),$$

and

$$E_1(t) = \int_{-\infty}^{+\infty} dt\, \beta_0(t) \left\| \sigma_B^{\frac{-1+it}{2}}\, \sigma_{AB}^{\frac{1-it}{2}}\, \sigma_A^{\frac{-1+it}{2}} - \mathbb{1}_{AB} \right\|_\infty \left\| \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+it}{2}}\, \sigma_B^{-1/2} \right\|_\infty,$$

$$E_2(t) = \int_{-\infty}^{+\infty} dt\, \beta_0(t) \left\| \sigma_B^{\frac{-1-it}{2}}\, \sigma_{AB}^{\frac{1+it}{2}}\, \sigma_A^{\frac{-1-it}{2}} - \mathbb{1}_{AB} \right\|_\infty.$$

Note that $\xi(\sigma_{AB}) = 0$ if σ_{AB} is a tensor product between A and B.



Relation with the classical case

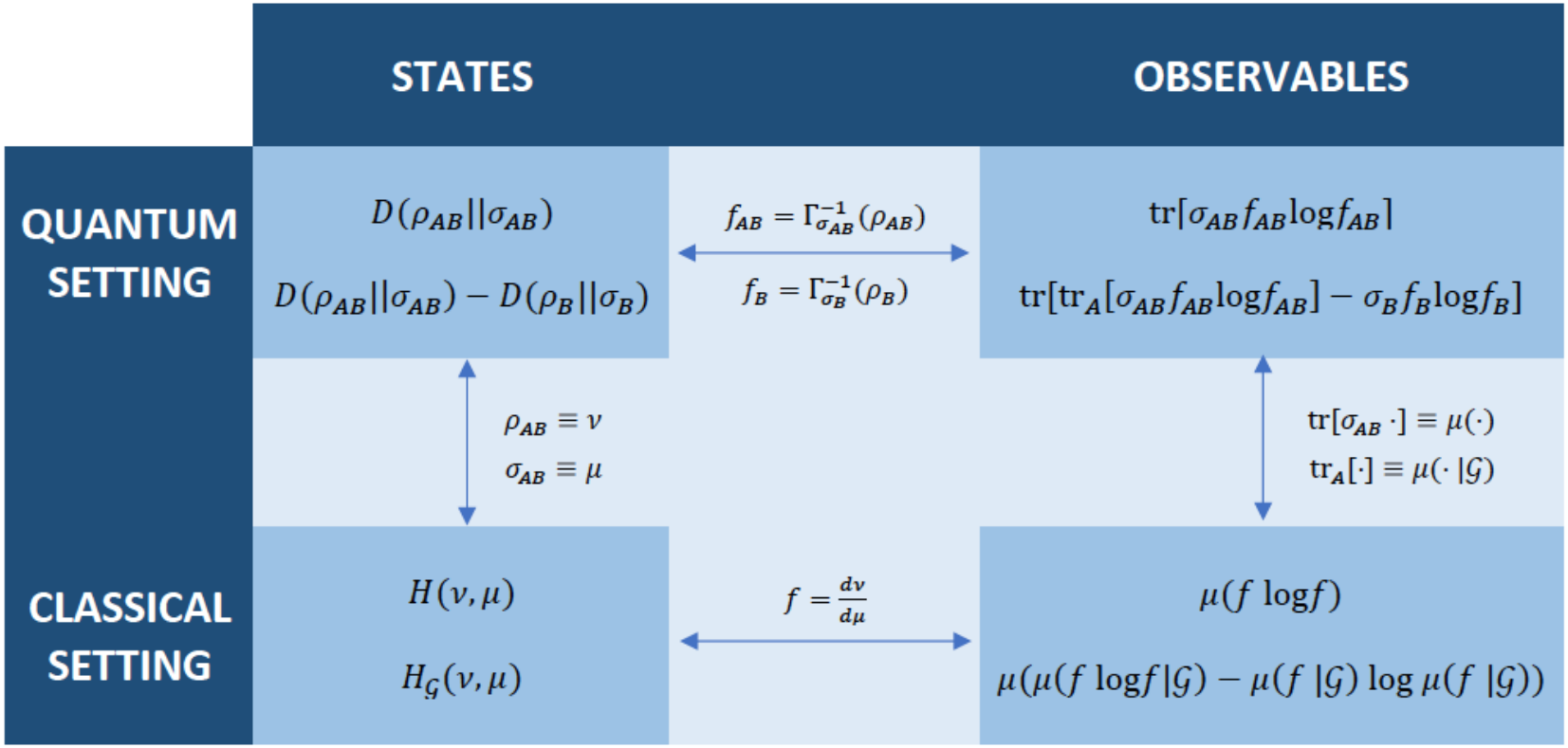
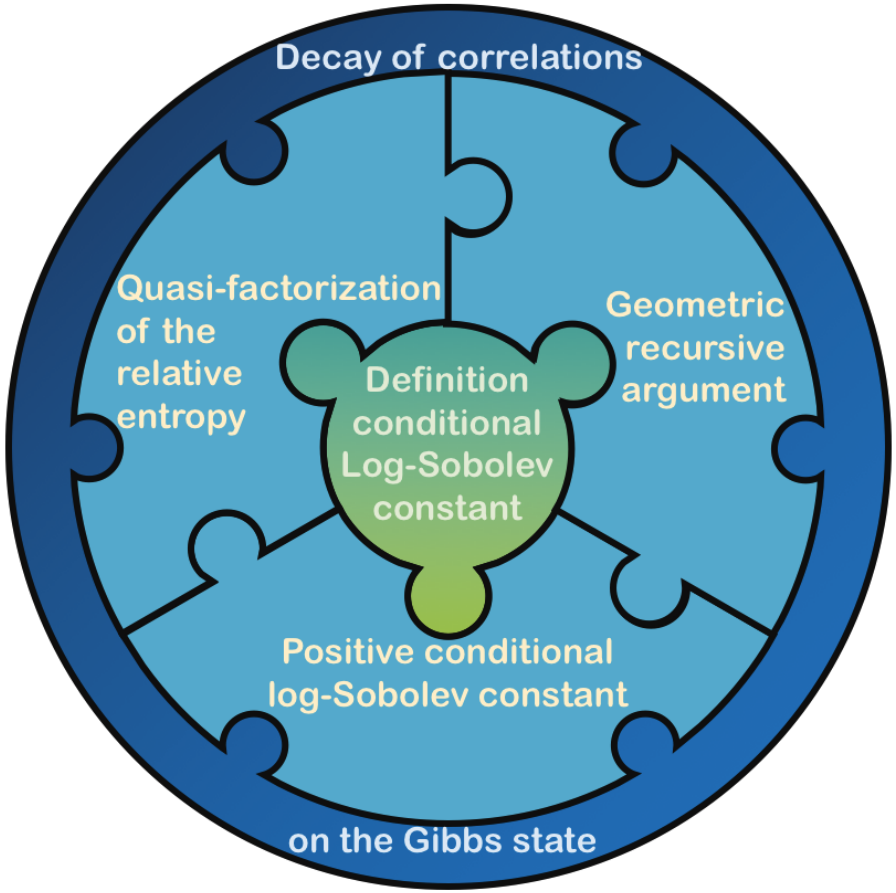


Figure: Identification between classical and quantum quantities when the states considered are classical.

2.3 Part 3: Log-Sobolev constants



Quantum spin lattices

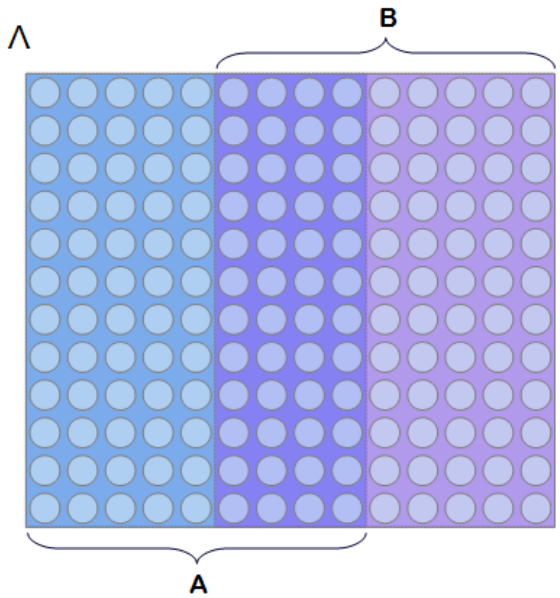


Figure: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

Problem

For a certain $\mathcal{L}^*_\Lambda$, can we prove $\alpha(\mathcal{L}^*_\Lambda) > 0$ using the result of quasi-factorization of the relative entropy?

Example 1 **(Q-Fact)**

Heat-bath dynamics with tensor product fixed point

Heat-bath with tensor product fixed point

Theorem (Q-Fact)

The **heat-bath dynamics**, with tensor product fixed point, has a positive log-Sobolev constant.

Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \ \mathcal{L}_\Lambda^* = \sum_{x\in\Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x\in\Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

Heat-bath with tensor product fixed point

Theorem (Q-Fact)

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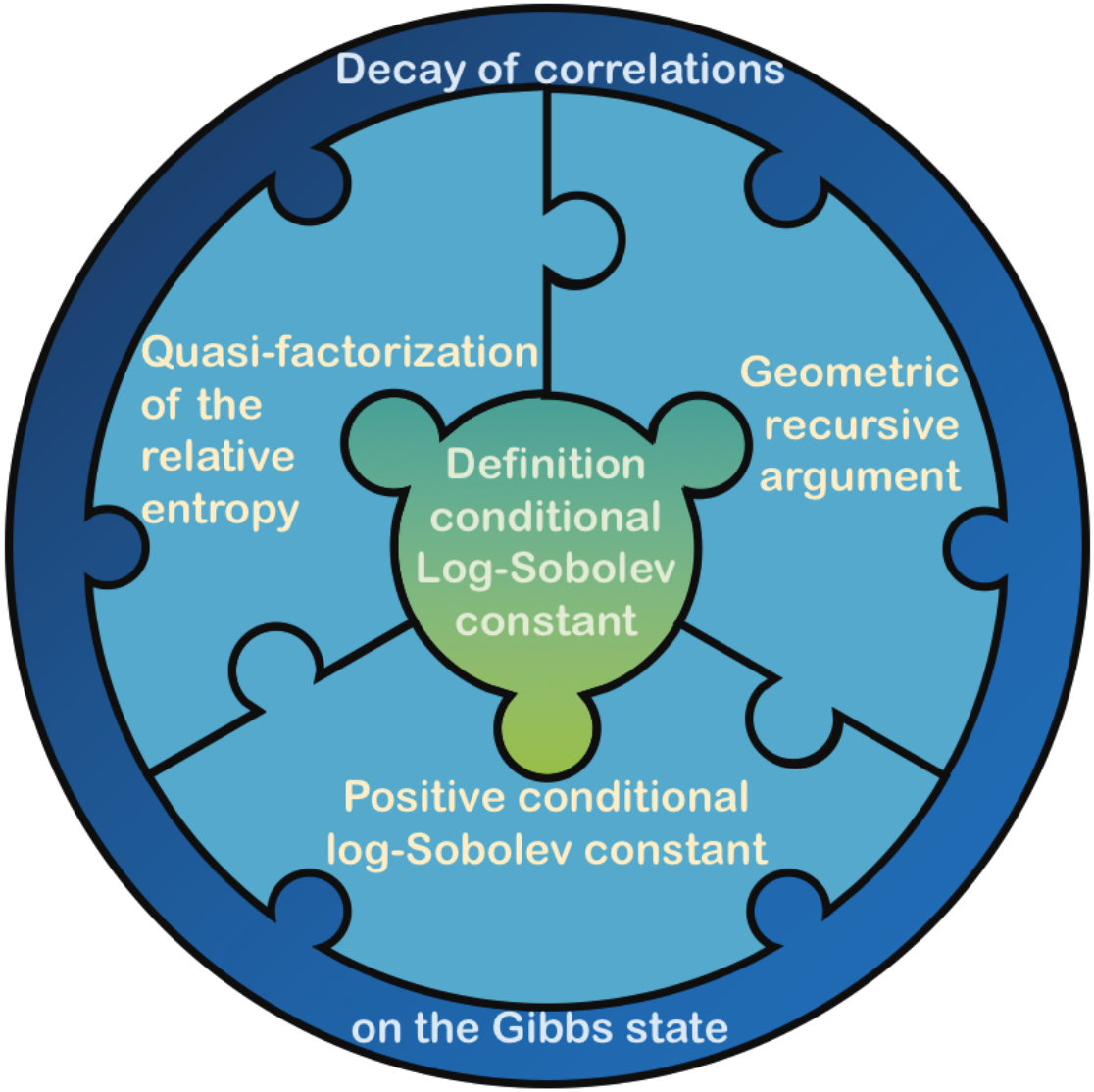
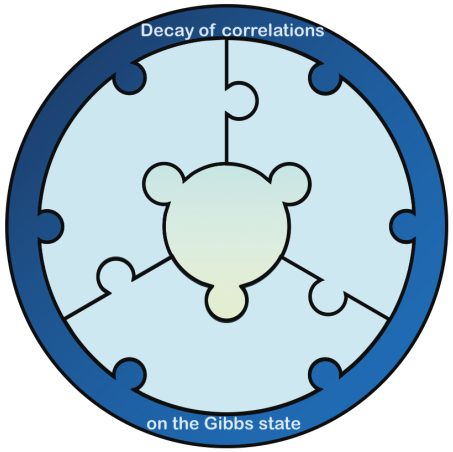
Strategy


Decay of correlations
Quasi-factorization of the relative entropy
Geometric recursive argument
Definition conditional Log-Sobolev constant
Positive conditional log-Sobolev constant
on the Gibbs state

Heat-bath with tensor product fixed point

Assumption

$$\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x .$$



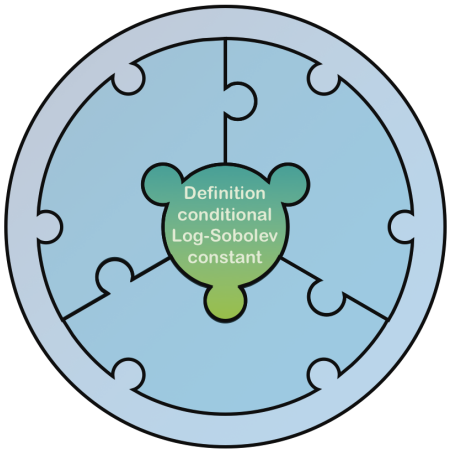
Heat-bath with tensor product fixed point

Conditional log-Sobolev constant

For $x \in \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in x by

$$\alpha_\Lambda(\mathcal{L}_x^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_x(\rho_\Lambda||\sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and $D_x(\rho_\Lambda||\sigma_\Lambda)$ is the conditional relative entropy.

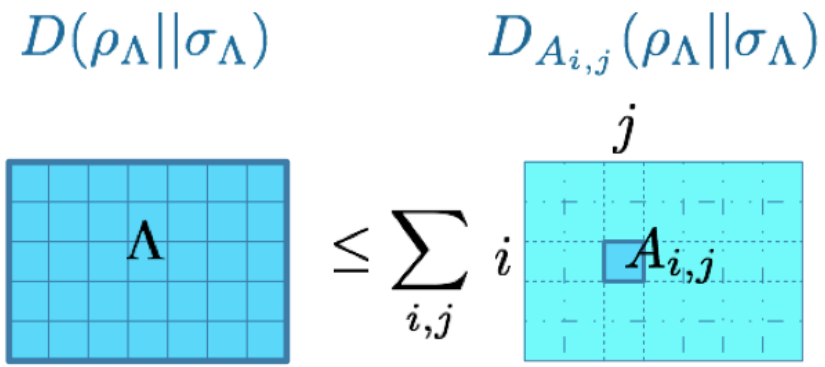


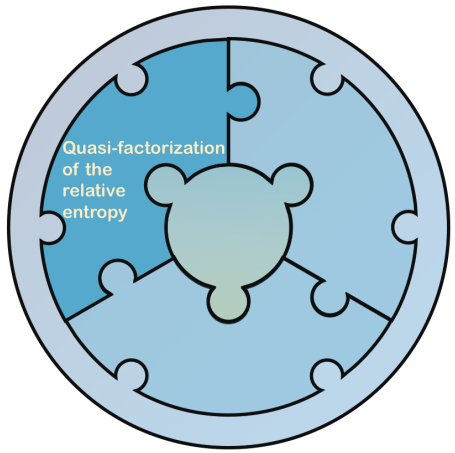
Heat-bath with tensor product fixed point

General quasi-factorization for σ a tensor product

Let $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ and $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x\in\Lambda} \sigma_x$. The following inequality holds:

$$D(\rho_\Lambda||\sigma_\Lambda) \leq \sum_{x\in\Lambda} D_x(\rho_\Lambda||\sigma_\Lambda).$$

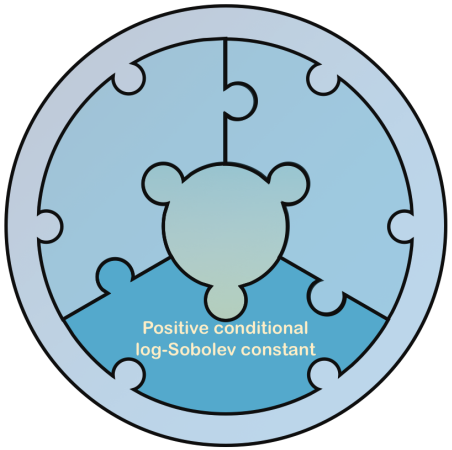




Heat-bath with tensor product fixed point

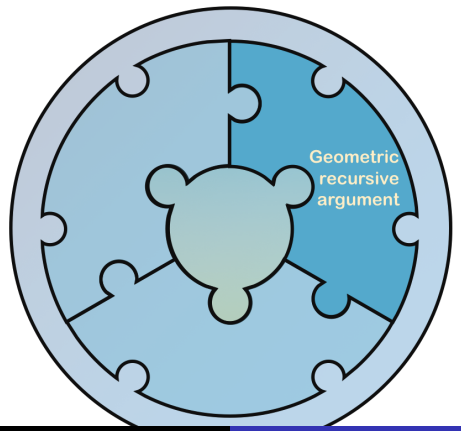
Lemma (Positivity of the conditional log-Sobolev constant)

$$\alpha_\Lambda(\mathcal{L}_x^*) \geq \frac{1}{2}.$$



Heat-bath with tensor product fixed point

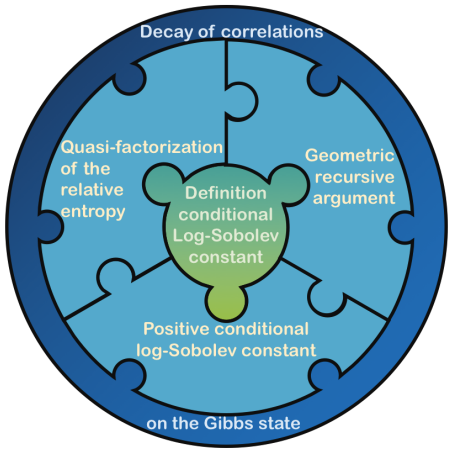
$$
\begin{aligned}
D(\rho_\Lambda||\sigma_\Lambda) &\leq \sum_{x\in\Lambda} D_x(\rho_\Lambda||\sigma_\Lambda) \\
&\leq \sum_{x\in\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_x^*)} \\
&\leq \frac{1}{2\inf\limits_{x\in\Lambda}\alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x\in\Lambda} -\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)] \\
&= \frac{1}{2\inf\limits_{x\in\Lambda}\alpha_\Lambda(\mathcal{L}_x^*)} \left(-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]\right) \\
&\leq \left(-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]\right).
\end{aligned}
$$



Heat-bath with tensor product fixed point

Positive log-Sobolev constant

$$\alpha(\mathcal{L}_\Lambda^*) \geq \frac{1}{2}.$$



Example 2, **(Heat-bath)**

Heat-bath dynamics in 1D

Heat-bath dynamics in 1D

σ_Λ is the Gibbs state of a k-local, commuting Hamiltonian.

$\Phi : \Lambda \to \mathcal{A}_\Lambda$ be a k-local potential: For $j \in \Lambda$, $\Phi(j)$ self-adjoint and supported on a ball of radius k around site j.

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Assume: $\|\Phi(j)\| \leq K$ for some constant $K < \infty$. The potential Φ is said to be **commuting** if for any $i, j \in \Lambda$, $[\Phi(i), \Phi(j)] = 0$.

Heat-bath dynamics in 1D

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Assume: $\|\Phi(j)\| \leq K$ for some constant $K < \infty$. The potential Φ is said to be **commuting** if for any $i, j \in \Lambda$, $[\Phi(i), \Phi(j)] = 0$.

Hamiltonian on a subregion $A \subseteq \Lambda$:

$$H_A := \sum_{j \in A} \Phi(j) \,. \tag{5}$$

Heat-bath dynamics in 1D

σ_Λ is the Gibbs state of a k-local, commuting Hamiltonian.

$\Phi : \Lambda \to \mathcal{A}_\Lambda$ be a k-local potential: For $j \in \Lambda$, $\Phi(j)$ self-adjoint and supported on a ball of radius k around site j.

Assume: $\|\Phi(j)\| \leq K$ for some constant $K < \infty$. The potential Φ is said to be **commuting** if for any $i, j \in \Lambda$, $[\Phi(i), \Phi(j)] = 0$.

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$$\sigma_A^\beta := \frac{\mathrm{e}^{-\beta H_A}}{\mathrm{tr}(\mathrm{e}^{-\beta H_A})} \,. \tag{6}$$

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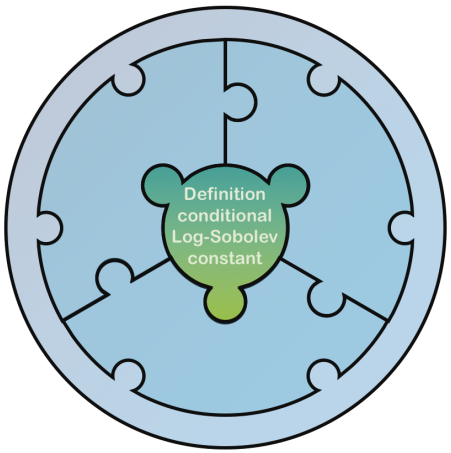
Conditional log-Sobolev constant

For $A \subset \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in A by

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda||\sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and

$$D_A(\rho_\Lambda||\sigma_\Lambda) = D(\rho_\Lambda||\sigma_\Lambda) - D(\rho_{A^c}||\sigma_{A^c}).$$



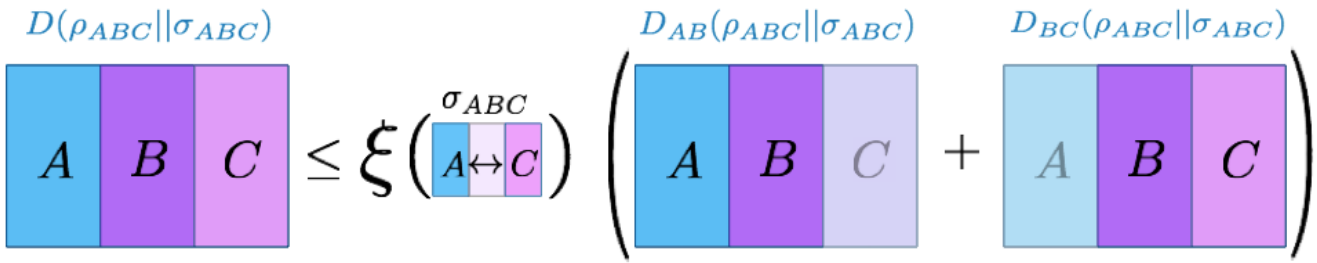
Quasi-factorization for the CRE (Q-Fact)

Let $\mathcal{H}_{ABC}$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. The following holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{AC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC})\right],$$

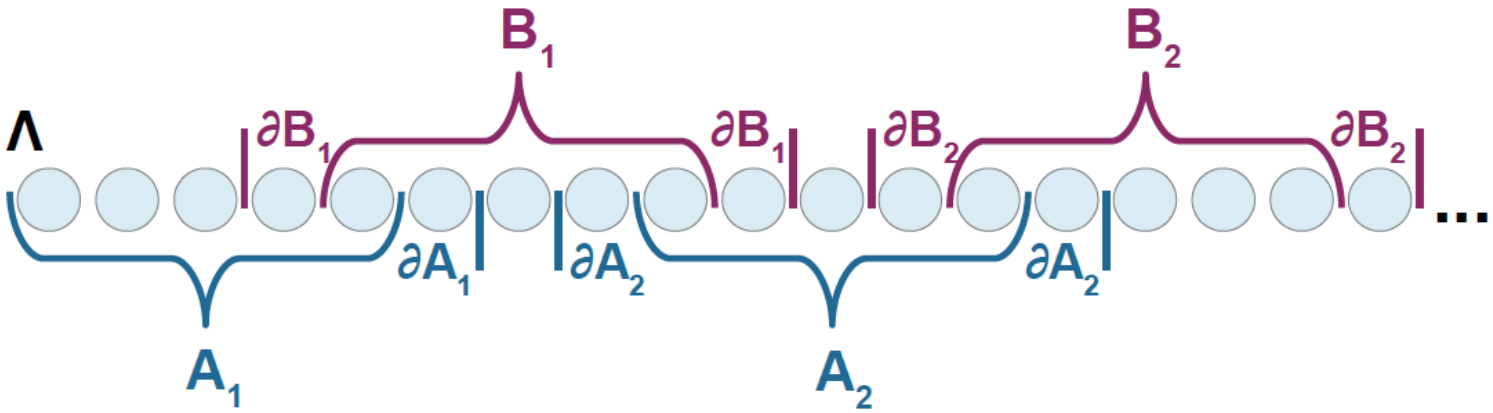
where

$$\xi(\sigma_{AC}) = \frac{1}{1 - 2\left\|\sigma_A^{-1/2} \otimes \sigma_C^{-1/2}\, \sigma_{AC}\, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}\right\|_\infty}.$$



Quasi-factorization of the relative entropy

STEP 1



$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{j=1}^{n} B_j$$

$$D(\rho_\Lambda||\sigma_\Lambda) \leq \frac{1}{1 - 2\|h(\sigma_{A^cB^c})\|_\infty} \left[D_A(\rho_\Lambda||\sigma_\Lambda) + D_B(\rho_\Lambda||\sigma_\Lambda)\right],$$

$$h(\sigma_{A^cB^c}) := \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^cB^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - \mathbb{1}_{A^cB^c}.$$

Quasi-factorization of the relative entropy

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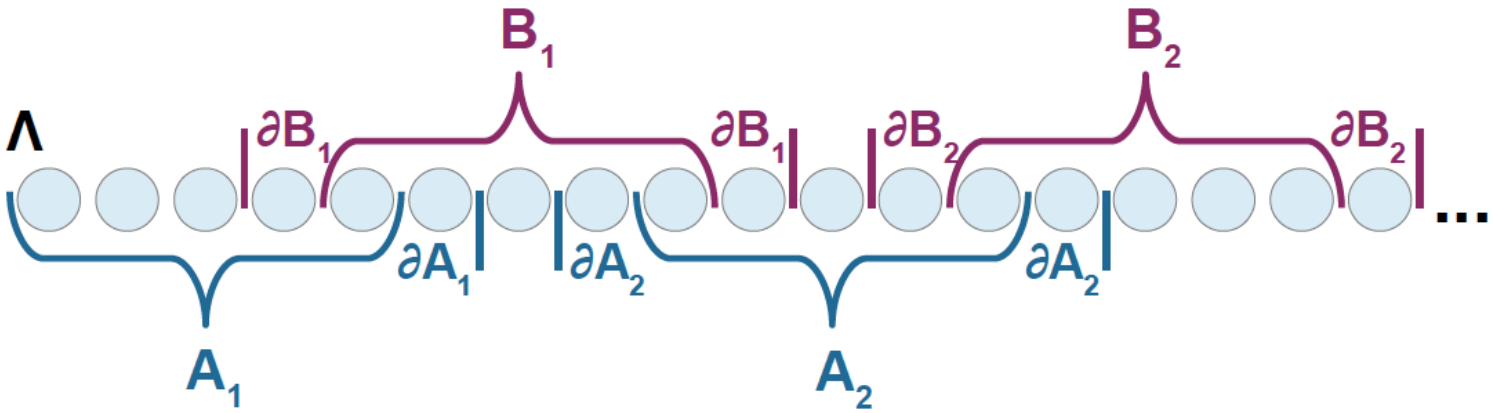

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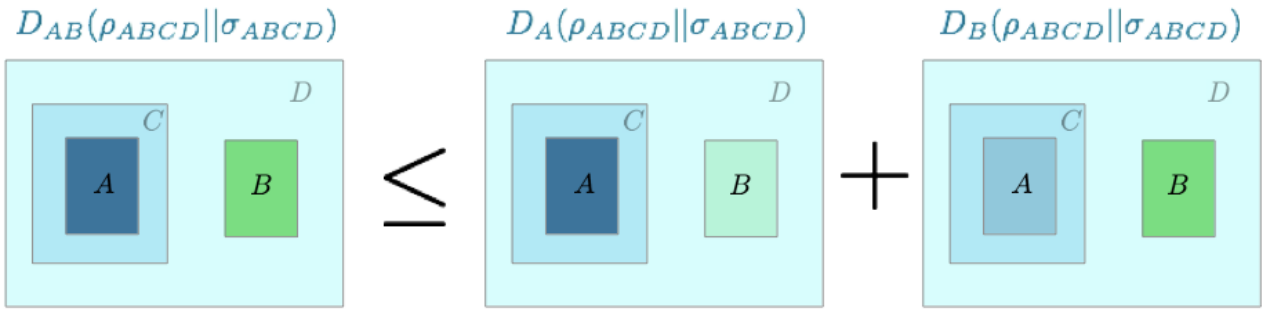
Quasi-factorization for QMC (Heat-bath)

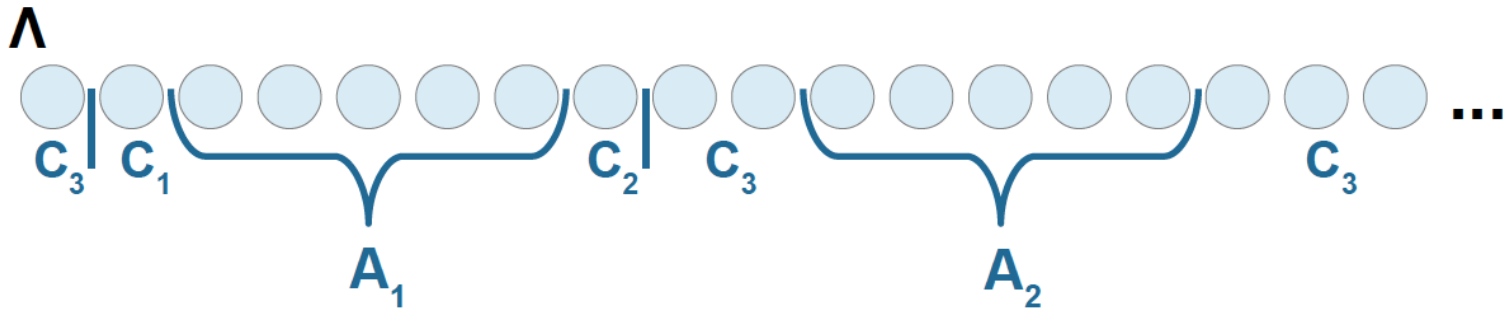
Let $\mathcal{H}_{ABCD} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$, where system C shields A from BD and $\rho_{ABCD}, \sigma_{ABCD} \in \mathcal{S}_{ABCD}$, such that σ_{ABCD} is a quantum Markov chain between $A \leftrightarrow C \leftrightarrow BD$. Then, the following holds

$$D_{AB}(\rho_{ABCD}||\sigma_{ABCD}) \leq [D_A(\rho_{ABCD}||\sigma_{ABCD}) + D_B(\rho_{ABCD}||\sigma_{ABCD})] .$$



Sketch of the proof

STEP 2



$$D_A(\rho_\Lambda||\sigma_\Lambda) \leq \sum_{i=1}^{n} D_{A_i}(\rho_\Lambda||\sigma_\Lambda)$$

σ_Λ is a QMC between $A_1 \leftrightarrow \partial A_1 \leftrightarrow \Lambda \setminus (A_1 \cup \partial A_1)$

$$\sigma_\Lambda = \bigoplus_{i\in I} \sigma_{A_1(\partial a_1)_i^L} \otimes \sigma_{(\partial a_1)_i^R \Lambda\setminus(A_1\cup\partial A_1)}$$

Sketch of the proof

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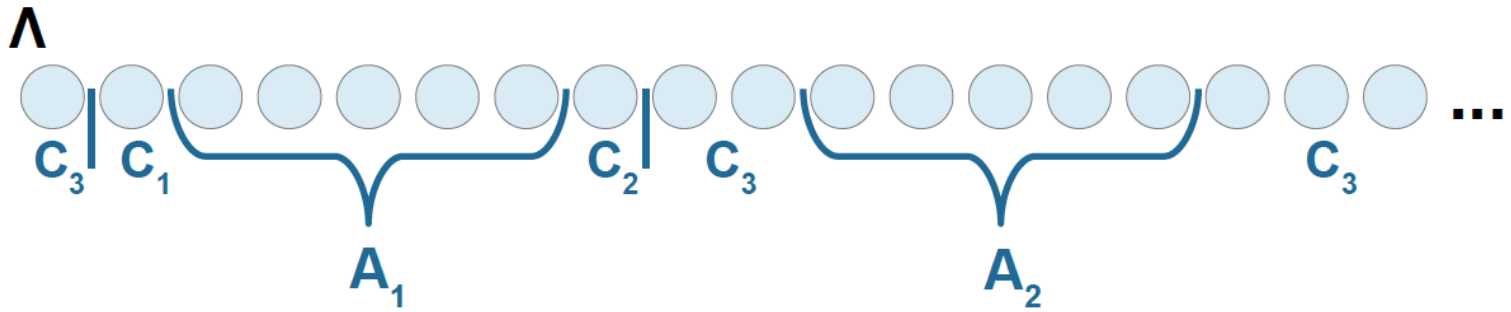

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Heat-bath dynamics in 1D

Assumption 1

In a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$, A and B not connected, we have

$$\|h(\sigma_{AB})\|_\infty = \left\|\sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB}\right\|_\infty \leq K < \frac{1}{2}.$$

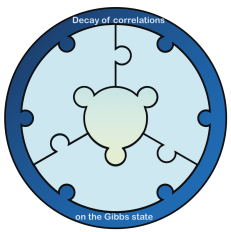
In particular, classical Gibbs states satisfy this.

Assumption 2

For any $B \subset \Lambda$, $B = B_1 \cup B_2$, it holds:

$$D_B(\rho_\Lambda || \sigma_\Lambda) \leq f(\sigma_{B\partial}) \left(D_{B_1}(\rho_\Lambda || \sigma_\Lambda) + D_{B_2}(\rho_\Lambda || \sigma_\Lambda)\right).$$

In particular, tensor products satisfy this (with $f = 1$).



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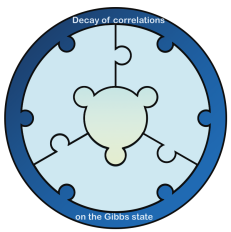
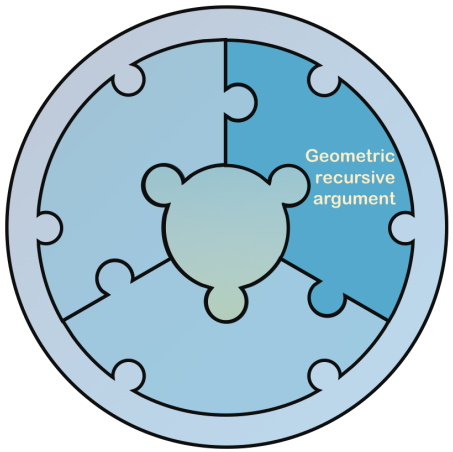


Heat-bath dynamics in 1D

STEP 3

$$\text{Assumption 1} \Rightarrow \alpha(\mathcal{L}^*_\Lambda) \geq \tilde{K} \min_{i\in\{1,\dots n\}} \left\{\alpha_\Lambda(\mathcal{L}^*_{A_i}), \alpha_\Lambda(\mathcal{L}^*_{B_i})\right\}$$



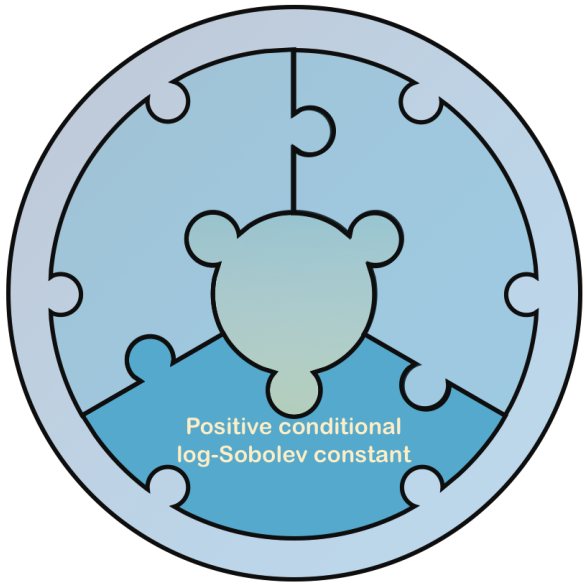
Using locality of the Lindbladian

$$\mathcal{L}^*_A + \mathcal{L}^*_B = \mathcal{L}^*_{A\cup B} + \mathcal{L}^*_{A\cap B}.$$

Heat-bath dynamics in 1D

STEP 4

Assumption 2 $\Rightarrow \alpha_\Lambda(\mathcal{L}^*_{A_i}) \geq g(\sigma_{A_i\partial}) > 0.$



Heat-bath dynamics in 1D

Theorem (Heat-bath)

In 1D, if Assumptions 1 and 2 hold, for a k-local commuting Hamiltonian, the heat-bath dynamics has a positive log-Sobolev constant.

Example 3 (Davies)

Davies dynamics

Davies dynamics

Generator

The generator of the Davies dynamics is of the following form:

$$\mathcal{L}_\Lambda^\beta(X) = i[H_\Lambda, X] + \sum_{k\in\Lambda} \mathcal{L}_k^\beta(X)\,,$$

where

$$\mathcal{L}_k^\beta(X) = \sum_{\omega,\alpha} \chi_{\alpha,k}^\beta(\omega) \left(S_{\alpha,k}^*(\omega) X S_{\alpha,k}(\omega) - \frac{1}{2}\left\{ S_{\alpha,k}^*(\omega) S_{\alpha,k}(\omega), X \right\} \right) .$$

Important property: Given $A \subseteq \Lambda$,

$$\mathcal{E}_A^\beta(X) := \mathcal{E}(X|\mathcal{N}) = \lim_{t\to\infty} \mathrm{e}^{t\mathcal{L}_A^\beta}(X)\,.$$

is a conditional expectation onto the subalgebra of fixed points of $\mathcal{L}_A^\beta$.

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Davies dynamics

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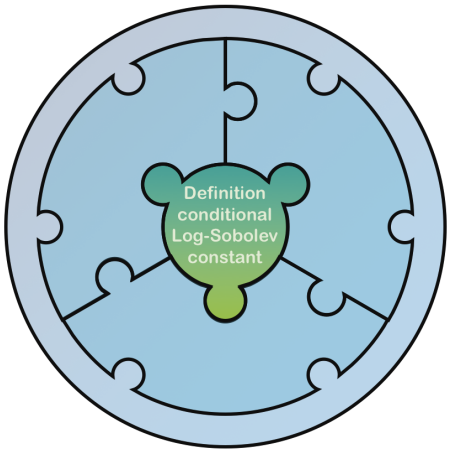
$$\alpha_\Lambda(\mathcal{L}_A^\beta) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}\left[\mathcal{L}_A^\beta(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)\right]}{2D_A^\beta(\rho_\Lambda||\sigma_\Lambda)},$$

where σ_Λ is the fixed point of the global evolution (the Gibbs state of a local commuting Hamiltonian), and

$$D_A^\beta(\rho_\Lambda||\sigma_\Lambda) = D(\rho_\Lambda||\mathcal{E}_A^\beta(\rho_\Lambda)).$$



Davies dynamics

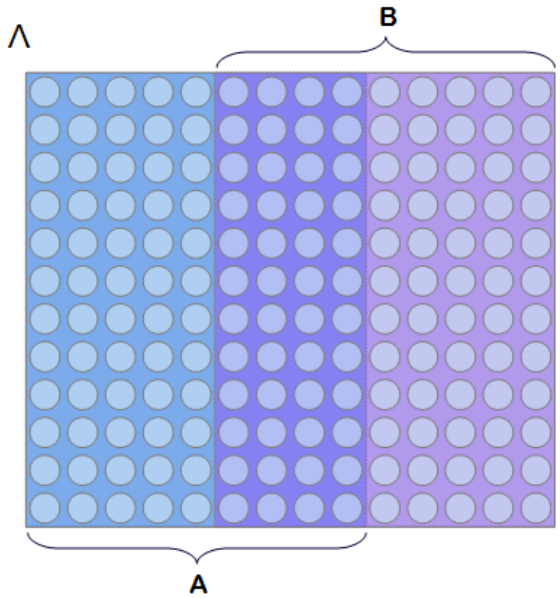


Figure: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

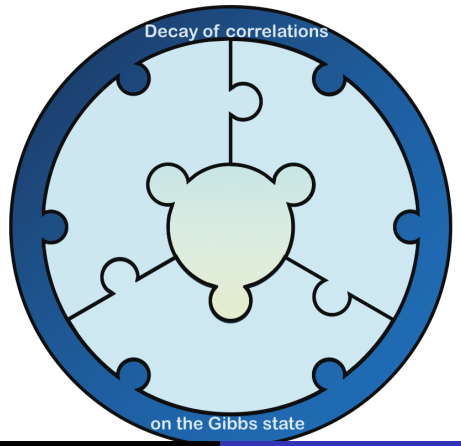
Davies dynamics

Clustering of correlations

The state $\sigma \in \mathcal{S}(\mathcal{H})$ is said to satisfy **exponential conditional $\mathbb{L}_1$-clustering of correlations** with respect to the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB})$ if there exists a constant $c := c(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB}, \sigma)$ such that, for any $X \in \mathcal{B}(\mathcal{H})$,

$$|\operatorname{Cov}_{\mathcal{N}_{AB},\sigma}(\mathcal{E}_A(X), \mathcal{E}_B(X))| \leq c \|X\|^2_{\mathbb{L}_1(\sigma)} e^{-d(A\setminus B, B\setminus A)/\xi}.$$

Moreover, the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB})$ is said to satisfy **exponential conditional $\mathbb{L}_1$-clustering of correlations** if there exists a constant $c := c(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB}, \sigma)$ such that any state $\sigma = \mathcal{E}^*_{AB}(\sigma)$ satisfies conditional $\mathbb{L}_1$-clustering of correlations with constant c.



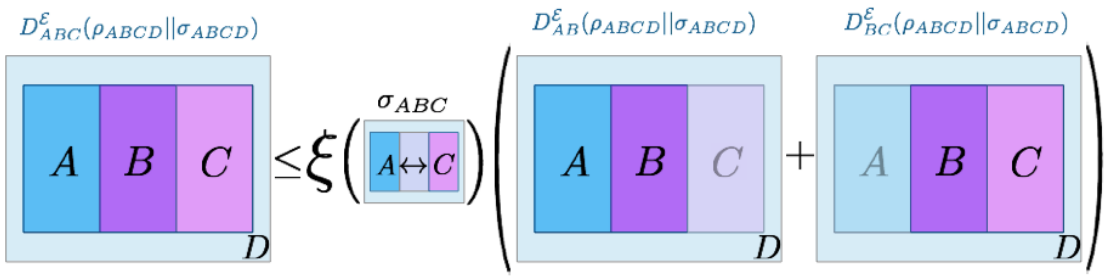
Davies dynamics

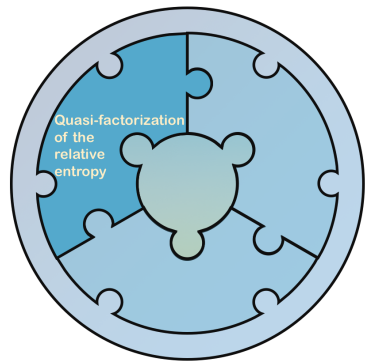
Quasi-factorization (Davies)

Assume that there exists a constant $0 < c < \dfrac{1}{2(4+\sqrt{2})}$ such that the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB})$ satisfies the exponential conditional $\mathbb{L}_1$-clustering of correlations with corresponding constant c. Then, the following inequality holds for every $\rho \in \mathcal{S}(\mathcal{H})$:

$$D^{\beta}_{AB}(\rho||\sigma) \leq \frac{1}{1-2(4+\sqrt{2})\,c}\left(D^{\beta}_{A}(\rho||\sigma) + D^{\beta}_{B}(\rho||\sigma)\right), \tag{7}$$

for every $\sigma = \mathcal{E}^{*}_{AB}(\sigma)$.





Geometric recursive argument (Davies)

$$\alpha\left(\mathcal{L}_{\Lambda}^{\beta *}\right) \geq \Psi(L_0) \min_{R \in \mathcal{R}_{L_0}} \alpha_{\Lambda}\left(\mathcal{L}_{R}^{\beta^*}\right),$$

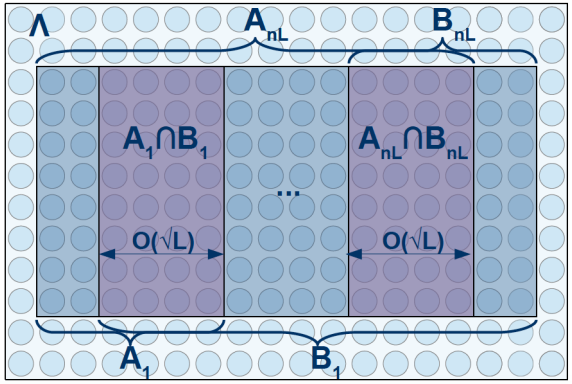
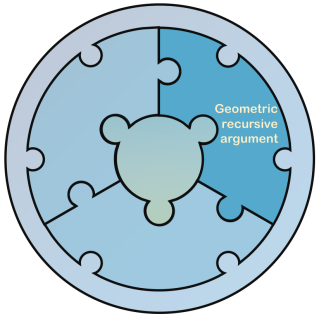


Figure: Splitting in A_n and B_n.

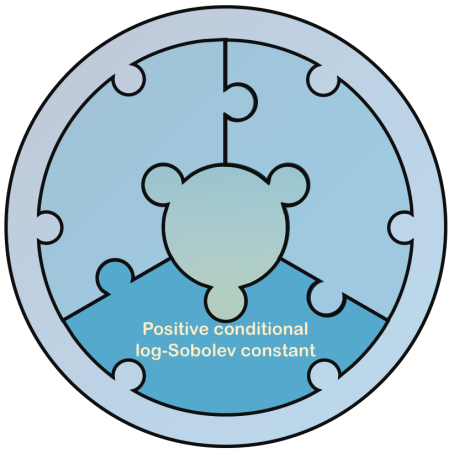


Theorem, Junge-LaRacuente-Rouzé '19

Given $\Lambda \subset\subset \mathbb{Z}^d$, $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ the Lindbladian associated to the Davies dynamics and a finite lattice and $A \subset \Lambda$, we have

$$\alpha_\Lambda \left(\mathcal{L}_A^{\beta *} \right) \geq \psi(|A|) > 0,$$

where $\psi(|A|)$ might depend on Λ, but is independent of its size.



2.4 Part 4: A strengthened DPI for the BS-entropy

Main concepts

Relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **relative entropy** is defined as:

$$D(\sigma||\rho) := \mathrm{tr}[\sigma(\log\sigma - \log\rho)].$$

Belavkin-Staszewski relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **BS-entropy** is defined as:

$$D_{\mathrm{BS}}(\sigma||\rho) := \mathrm{tr}\left[\sigma \log\left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right].$$

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Relation between relative entropies

The following holds for every $\sigma > 0, \rho > 0$:

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Motivation: Relative entropy

Relative entropy of σ and ρ: $D(\sigma||\rho) := \mathrm{tr}[\sigma(\log\sigma - \log\rho)]$.

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$$D(\sigma||\rho) \geq D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)).$$

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Conditions for equality

$$D(\sigma||\rho) = D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \Leftrightarrow \sigma = \rho^{1/2}\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1/2}\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1/2}\right)\rho^{1/2}.$$

Petz recovery map $\mathcal{R}^{\rho}_{\mathcal{T}}(\cdot) := \rho^{1/2}\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1/2}(\cdot)\mathcal{T}(\rho)^{-1/2}\right)\rho^{1/2}$.

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Motivation: Strengthened bounds for DPI of RE

Problem

Can we find a lower bound for the DPI in terms of $D(\sigma||\mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma))$*?*

Answer: It is not possible **(Brandao et al. '15, Fawzi[2] '17)**.

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(Carlen-Vershynina '17) $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ conditional expectation, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$:

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Standard and maximal f-divergences

(Hiai-Mosonyi '17)

Standard f-divergences

Let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$S_f(\sigma \| \rho) = \mathrm{tr}\left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) \rho^{1/2}\right]$$

is the *standard f-divergence*.

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Let $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a PTP linear map. Then,

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Relation between f-divergences

For every two states $\sigma > 0$, $\rho > 0$ on $\mathcal{M}$ and every operator convex function $f : (0, \infty) \to \mathbb{R}$,

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Equivalent conditions for equality on DPI

$\Gamma := \sigma^{-1/2}\rho\sigma^{-1/2}$ and $\Gamma_{\mathcal{N}} := \sigma_{\mathcal{N}}^{-1/2}\rho_{\mathcal{N}}\sigma_{\mathcal{N}}^{-1/2}$
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Equivalent conditions for equality on DPI (BS-entropy)

Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on $\mathcal{M}$. Then, the following are equivalent:

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Note: Although they can be seen as a consequence of the previous result, the following facts were previously known.

Corollary

$$\begin{aligned}\hat{S}_{\mathrm{BS}}(\sigma\|\rho) = \hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) &\Leftrightarrow \rho = \mathcal{T}_{\mathcal{E}}^{\sigma}\circ\mathcal{E}(\rho)\\ &\Leftrightarrow \sigma = \mathcal{T}_{\mathcal{E}}^{\rho}\circ\mathcal{E}(\sigma)\\ &\Leftrightarrow \hat{S}_{\mathrm{BS}}(\rho\|\sigma) = \hat{S}_{\mathrm{BS}}(\rho_{\mathcal{N}}\|\sigma_{\mathcal{N}}).\end{aligned}$$

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$$D(\sigma\|\rho) = D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \implies \hat{S}_{BS}(\sigma\|\rho) = \hat{S}_{BS}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}).$$

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$$\hat{S}_{\mathrm{BS}}(\sigma\|\rho)-\hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}})\geq\left(\frac{\pi}{8}\right)^{4}\|\Gamma\|_{\infty}^{-4}\left\|\sigma^{-1}\right\|_{\infty}^{-2}\left\|\rho-\sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\right\|_{2}^{4}.$$

Strengthened DPI for maximal f-divergences

Strengthened DPI for maximal f-divergences **(BS-entropy)**

Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E}:\mathcal{M}\to\mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma>0$, $\rho>0$ be two quantum states on $\mathcal{M}$ and let $f:(0,\infty)\to\mathbb{R}$ be an operator convex function with transpose $\tilde{f}$. We assume that $\tilde{f}$ is operator monotone decreasing and such that the measure $\mu_{-\tilde{f}}$ that appears in the representation of $-\tilde{f}$ is absolutely continuous with respect to Lebesgue measure. Moreover, we assume that for every $T\geq 1$, there exist constants $\alpha\geq 0$, $C>0$ satisfying $\mathrm{d}\mu_{-\tilde{f}}(t)/\mathrm{d}t\geq (CT^{2\alpha})^{-1}$ for all $t\in[1/T,T]$ and such that

$$\left(\frac{(2\alpha+1)\sqrt{C}}{4}\,\frac{(\hat{S}_f(\sigma\|\rho)-\hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}))^{1/2}}{1+\|\Gamma\|_\infty}\right)^{\frac{1}{1+\alpha}}\leq 1.$$

Then, there is a constant $L_\alpha>0$ such that

$$\begin{aligned}&\hat{S}_f(\sigma\|\rho)-\hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}})\geq\\&\geq\frac{L_\alpha}{C}\left(1+\|\Gamma\|_\infty\right)^{-(4\alpha+2)}\|\Gamma\|_\infty^{-(2\alpha+2)}\left\|\sigma^{-1}\right\|_\infty^{-(2\alpha+2)}\left\|\rho-\sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\right\|_2^{4(\alpha+1)}.\end{aligned}$$

thank you
danke
謝謝
ngiyabonga
شكراً جزيلاً
teşekkür ederim
gracias
спасибо
Баярлалаа
рахмат
dank je
tapadh leat
хвала
bedankt
dziękuję
obrigado
sukriya
kop khun krap
go raibh maith agat
mochchakkeram
arigatô
dakujem
grazie
terima kasih
감사합니다
ありがとう
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gràçie
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didi madloba
xièxie
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