The modified logarithmic Sobolev inequality for quantum spin systems via approximate tensorization of the relative entropy

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(and Ivan Bardet, Andreas Bluhm, Angelo Lucia, David Pérez-García and Antonio Pérez-Hernández)

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Open quantum systems

Problem

Velocity of convergence of certain quantum dissipative evolutions to their thermal equilibriums.

No experiment can be executed at zero temperature or be completely shielded from noise.

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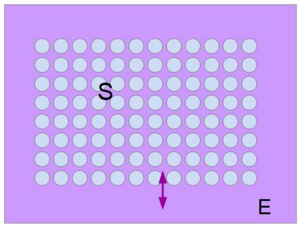
⇒ Open quantum many-body systems.



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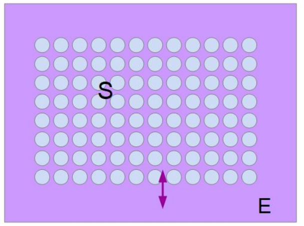

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- The continuous-time evolution of a state on S is given by a q. Markov semigroup (**Markovian approximation**).

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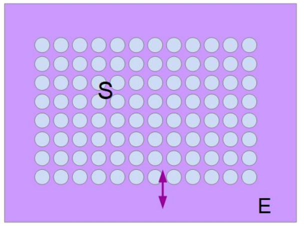

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Quantum Markov semigroups

Quantum Markov semigroups

A **quantum Markov semigroup** is a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t^*(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$

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Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\|\rho_t - \sigma_\Lambda\right\|_1 \leq \text{poly}(|\Lambda|)e^{-\gamma t}.$$

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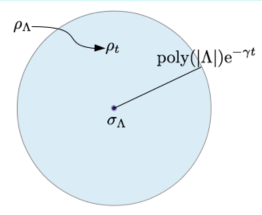

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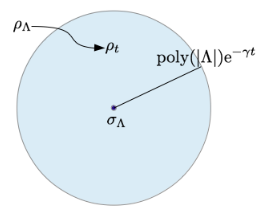

Modified logarithmic Sobolev inequality

MLSI constant

The **MLSI constant** of $\mathcal{L}_\Lambda^*$ is defined as:

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

If $\liminf_{\Lambda \nearrow \mathbb{Z}^d} \alpha(\mathcal{L}_\Lambda^*) > 0$:

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$$\left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \sqrt{2D(\rho_\Lambda || \sigma_\Lambda)}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t} \leq \sqrt{2\log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t}.$$

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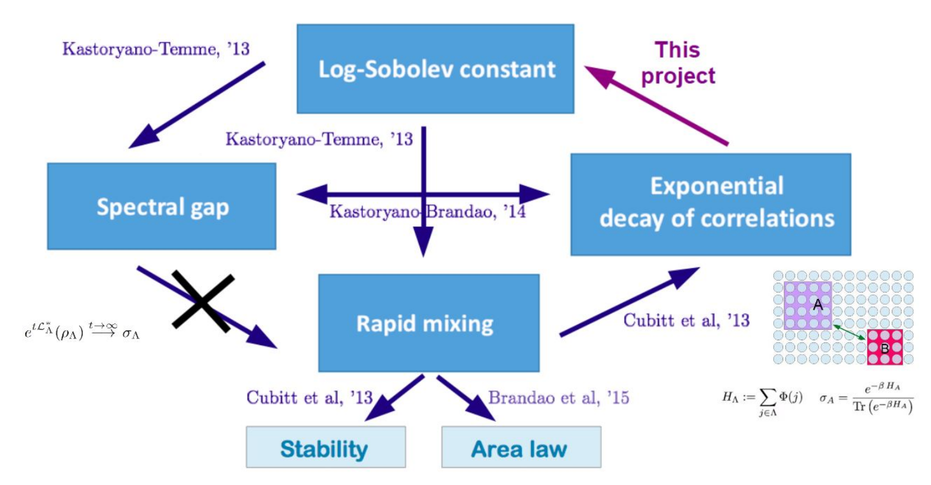
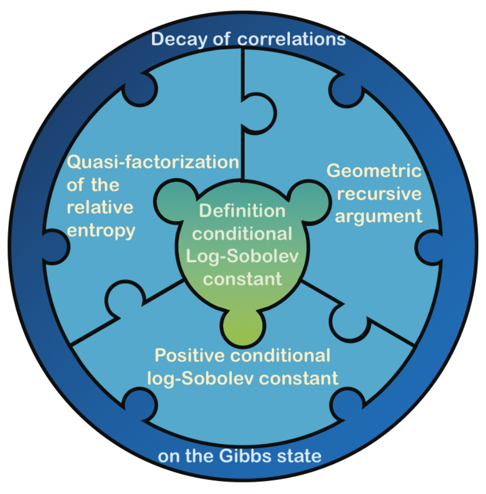
Quantum spin systems

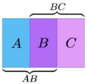


Strategy

Used in (C.-Lucia-Pérez García '18) and (Bardet-C.-Lucia-Pérez García-Rouzé, '19).

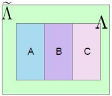


Quasi-factorization / Approximate tensorization



Quasi-factorization / Approximate tensorization of the relative entropy $\quad \Lambda = ABC$

$$D(\rho_\Lambda \| \sigma_\Lambda) \leq c \left[D_{AB}(\rho_\Lambda \| \sigma_\Lambda) + D_{BC}(\rho_\Lambda \| \sigma_\Lambda) \right] + d$$



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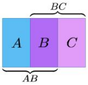

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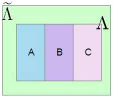
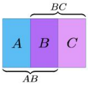


Classical quasi-factorization

$$\mathrm{Ent}(f) \leq c\, \mu\, [\mathrm{Ent}(f|\mathcal{F}_1) + \mathrm{Ent}(f|\mathcal{F}_2)]$$

Cesi02,
DPP02

Quasi-factorization / Approximate tensorization



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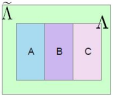

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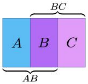
Cesi02, DPP02

Strong subadditivity

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$$

LR73

Quasi-factorization / Approximate tensorization



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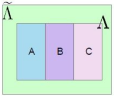

Classical quasi-factorization (Cesi02, DPP02)

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$$D_A(\rho_\Lambda \| \sigma_\Lambda) := D(\rho_\Lambda \| \sigma_\Lambda) - D(\rho_{A^c} \| \sigma_{A^c})$$

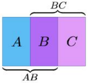
CLP18

Quantum quasi-factorization

$$D(\Lambda) \leq \frac{1}{1 - 2\| H(\sigma_\Lambda) \|_\infty} \left[D_{AB}(\Lambda) + D_{BC}(\Lambda) \right]$$

$$H(\sigma_\Lambda) := \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \sigma_{AC} \sigma_A^{-1/2} \otimes \sigma_C^{-1/2}$$

Quasi-factorization / Approximate tensorization



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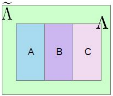

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CLP18

General superadditivity

CLP18'

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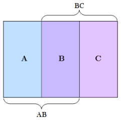
Quasi-factorization for the relative entropy



$$D_A(\rho_{ABC}||\sigma_{ABC}) := D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_{BC}||\sigma_{BC})$$

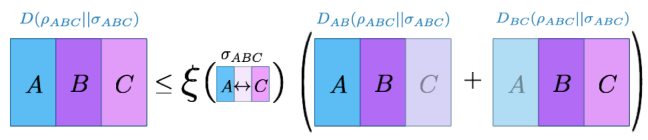
Quasi-factorization for the CRE (C.-Lucia-Pérez García '18)

Let $\mathcal{H}_{ABC}$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. The following holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{AC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where

$$\xi(\sigma_{AC}) = \frac{1}{1 - 2\left\| \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \, \sigma_{AC} \, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC} \right\|_\infty}.$$



General superadditivity for the relative entropy

$$\begin{aligned}(1-2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) &\leq D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \\ &= 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).\end{aligned}$$

$$\Leftrightarrow$$

$$(1+2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

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The previous result is equivalent to (C.-Lucia-Pérez García '18):

$$\boxed{(1 + 2\|H(\sigma_{AB})\|_\infty) D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

General superadditivity for the relative entropy

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Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

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Recall:

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Due to:

- **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

we have

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

General superadditivity for the relative entropy

$$
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$$\boxed{(1 + 2\|H(\sigma_{AB})\|_\infty) D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

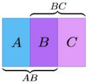
Due to:

- **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

we have

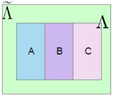
$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

Quasi-factorization / Approximate tensorization



Quasi-factorization / Approximate tensorization of the relative entropy $\Lambda = ABC$

$$D(\rho_\Lambda \| \sigma_\Lambda) \leq c\,[D_{AB}(\rho_\Lambda\|\sigma_\Lambda) + D_{BC}(\rho_\Lambda\|\sigma_\Lambda)] + d$$



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$$D_A(\rho_\Lambda\|\sigma_\Lambda) := D(\rho_\Lambda\|\sigma_\Lambda) - D(\rho_{A^c}\|\sigma_{A^c})$$

CLP18

General superadditivity

CLP18'

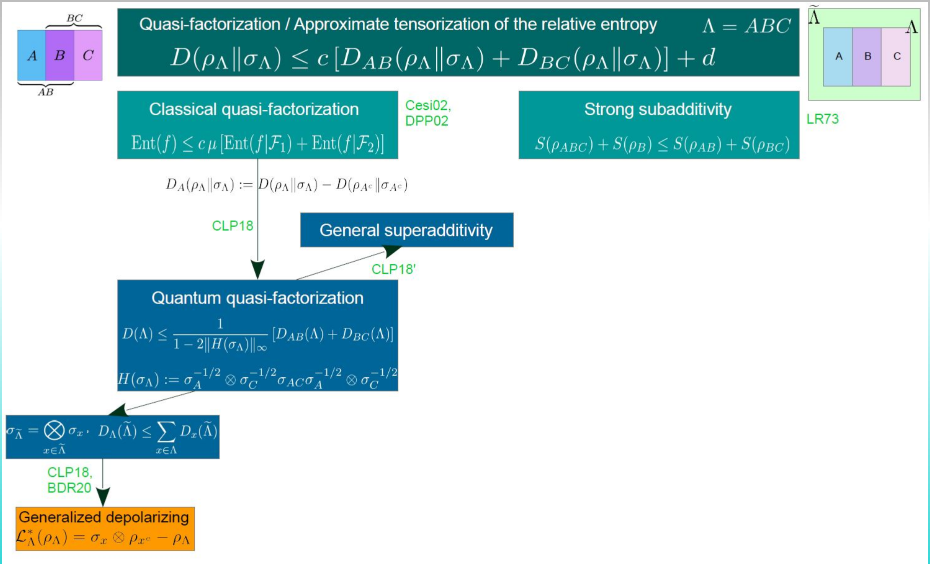
Quantum quasi-factorization

$$D(\Lambda) \leq \frac{1}{1 - 2\|H(\sigma_\Lambda)\|_\infty}\,[D_{AB}(\Lambda) + D_{BC}(\Lambda)]$$

$$H(\sigma_\Lambda) := \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \sigma_{AC} \sigma_A^{-1/2} \otimes \sigma_C^{-1/2}$$

$$\sigma_{\widetilde{\Lambda}} = \bigotimes_{x \in \widetilde{\Lambda}} \sigma_x, \quad D_\Lambda(\widetilde{\Lambda}) \leq \sum_{x \in \Lambda} D_x(\widetilde{\Lambda})$$

Quasi-factorization / Approximate tensorization



Heat-bath with tensor product fixed point

Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \quad \mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

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Generalized depolarizing semigroup.

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Let $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ and $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. The following inequality holds:

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The **heat-bath dynamics**, with tensor product fixed point, has MLSI(1/2).

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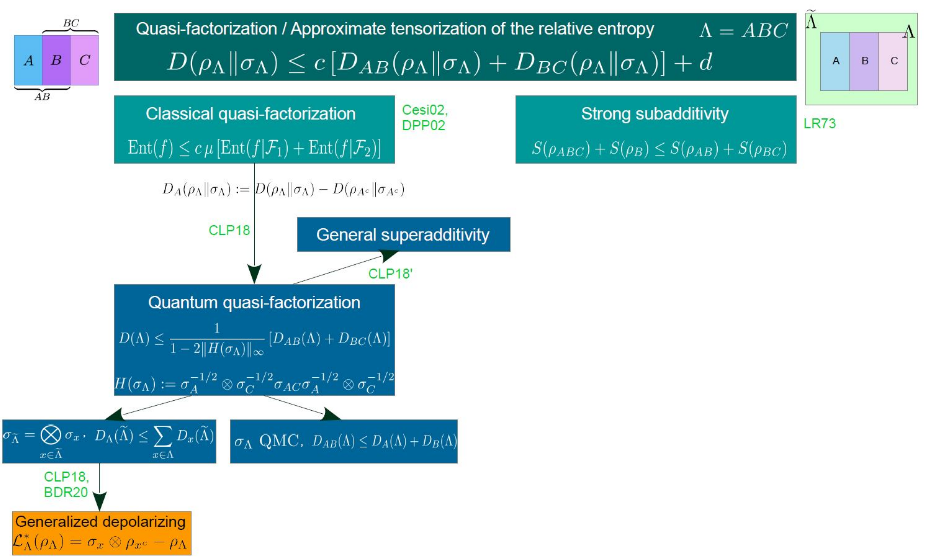

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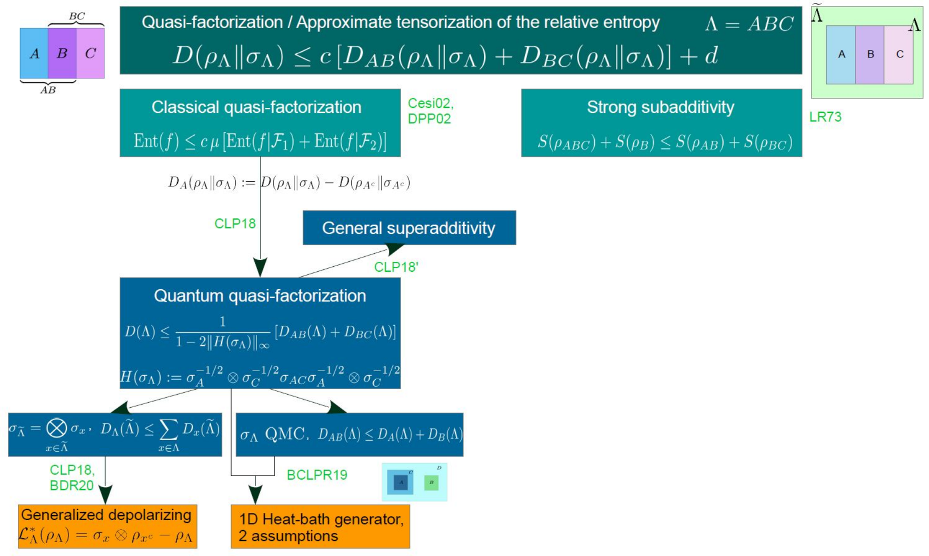
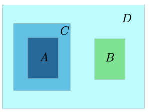


Heat-bath dynamics in 1D



$\sigma_\Lambda = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}$ is the Gibbs state of a k-local, commuting Hamiltonian H.

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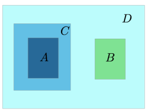

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Quasi-factorization for quantum Markov chains (Bardet-C.-Lucia-Pérez García-Rouzé'19)

Let $\mathcal{H}_{ABCD} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \otimes \mathcal{H}_D$, where C shields A from B and D, and let $\rho_{ABCD}, \sigma_{ABCD} \in \mathcal{S}_{ABCD}$. Assume that σ_{ABCD} is a QMC between $A \leftrightarrow C \leftrightarrow BD$. Then, the following inequality holds:

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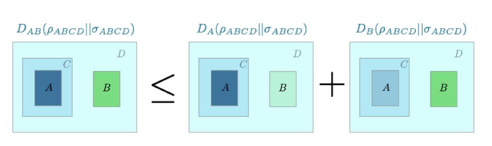

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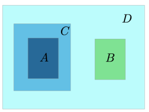

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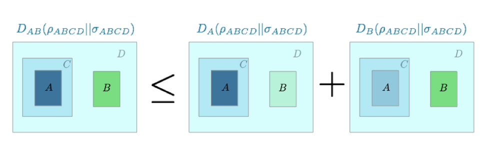

Heat-bath dynamics in 1D

Assumption 1

In a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$, A and B not connected, we have

$$\left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty \leq K < \frac{1}{2}.$$

In particular, Gibbs states at high enough temperature satisfy this.

Assumption 2

For any $B \subset \Lambda$, $B = B_1 \cup B_2$, it holds:

$$D_B(\rho_\Lambda || \sigma_\Lambda) \leq f(\sigma_{B\partial}) \left(D_{B_1}(\rho_\Lambda || \sigma_\Lambda) + D_{B_2}(\rho_\Lambda || \sigma_\Lambda) \right).$$

In particular, tensor products satisfy this (with $f = 1$).

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In 1D, if Assumptions 1 and 2 hold, for a k-local commuting Hamiltonian, the heat-bath dynamics has a positive log-Sobolev constant.

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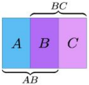

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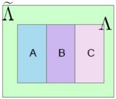

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$$D_A(\rho_\Lambda \| \sigma_\Lambda) := D(\rho_\Lambda \| \sigma_\Lambda) - D(\rho_{A^c} \| \sigma_{A^c})$$

BS-entropy

$$\widehat{D}(\rho\|\sigma) := \text{Tr}[\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2})]$$

$$\widehat{D}(\Lambda) \leq c \left[\widehat{D}_{AB}(\Lambda) + \widehat{D}_{BC}(\Lambda) \right] + d$$

CLP18

General superadditivity

CLP18'

BCP21

Quantum quasi-factorization

$$D(\Lambda) \leq \frac{1}{1 - 2\|H(\sigma_\Lambda)\|_\infty} \left[D_{AB}(\Lambda) + D_{BC}(\Lambda) \right]$$

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$$\sigma_\Lambda \text{ QMC}, \quad D_{AB}(\Lambda) \leq D_A(\Lambda) + D_B(\Lambda)$$

CLP18, BDR20

BCLPR19

Generalized depolarizing

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sigma_x \otimes \rho_{x^c} - \rho_\Lambda$$

1D Heat-bath generator, 2 assumptions

BS-entropy

$$\widehat{D}(\rho_{AB}||\sigma_{AB}) := \operatorname{tr}\left[\rho_{AB} \log\left(\rho_{AB}^{1/2}\,\sigma_{AB}^{-1}\,\rho_{AB}^{1/2}\right)\right], \quad \widehat{D}_A(\rho_{AB}||\sigma_{AB}) := \widehat{D}(\rho_{AB}||\sigma_{AB}) - \widehat{D}(\rho_B||\sigma_B).$$

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Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following inequality holds whenever $\|H(\sigma_{AB})\|_\infty < 1/2$:

$$\widehat{D}(\rho_{AB}||\sigma_{AB}) \leq \widetilde{M}(\sigma_{AB}) \left[\widehat{D}_A(\rho_{AB}||\sigma_{AB}) + \widehat{D}_B(\rho_{AB}||\sigma_{AB})\right] + \widetilde{L}(\rho_{AB}, \sigma_{AB}),$$

where

$$\widetilde{M}(\sigma_{AB}) := \frac{1}{1 - 2\|H(\sigma_{AB})\|_\infty},$$

and

$$\widetilde{L}(\rho_{AB}, \sigma_{AB}) \leq f\left(\left\|\left[\rho_A^{1/2}, \sigma_A^{-1/2}\right]\right\|_\infty, \left\|\left[\rho_B^{1/2}, \sigma_B^{-1/2}\right]\right\|_\infty\right).$$

Note that if $\sigma_{AB} = \sigma_A \otimes \sigma_B$, we have $\widetilde{M}(\sigma_{AB}) = 1$, and if $\rho_A^{1/2}\sigma_A^{-1/2}$ and $\rho_B^{1/2}\sigma_B^{-1/2}$ are normal (in particular, if $[\rho_A, \sigma_A] = [\rho_B, \sigma_B] = 0$), then $\widetilde{L}(\rho_{AB}, \sigma_{AB}) = 0$.

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BS-entropy

If $\widetilde{L}(\rho_{AB}, \sigma_{AB}) = 0$ in general, the previous result would be equivalent to superadditivity for the BS-entropy.

However, continuity, additivity, superadditivity and monotonicity characterize the **relative entropy** (Wilming et at. '17, Matsumoto '10).

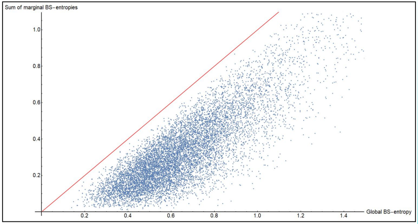
BS-entropy

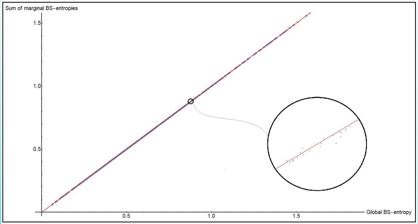
If $\widetilde{L}(\rho_{AB}, \sigma_{AB}) = 0$ in general, the previous result would be equivalent to superadditivity for the BS-entropy.

However, continuity, additivity, superadditivity and monotonicity characterize the **relative entropy** (Wilming et at. ’17, Matsumoto ’10).

We plot $\widehat{D}(\rho_{AB}||\sigma_{AB})$ against $\widehat{D}(\rho_A||\sigma_A) + \widehat{D}(\rho_B||\sigma_B)$ for

$$\rho_{AB} := \frac{\eta_A \otimes \eta_B + \varepsilon \lambda_{AB}}{\mathrm{tr}[\eta_A \otimes \eta_B + \varepsilon \lambda_{AB}]} .$$





BS-entropy

If $\widetilde{L}(\rho_{AB}, \sigma_{AB}) = 0$ in general, the previous result would be equivalent to superadditivity for the BS-entropy.

However, continuity, additivity, superadditivity and monotonicity characterize the **relative entropy** (Wilming et at. '17, Matsumoto '10).

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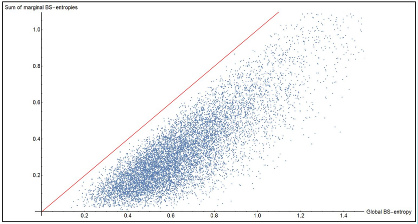
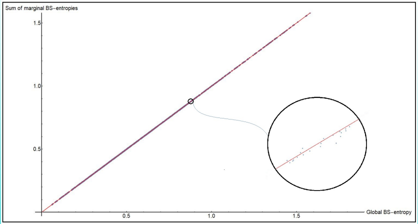
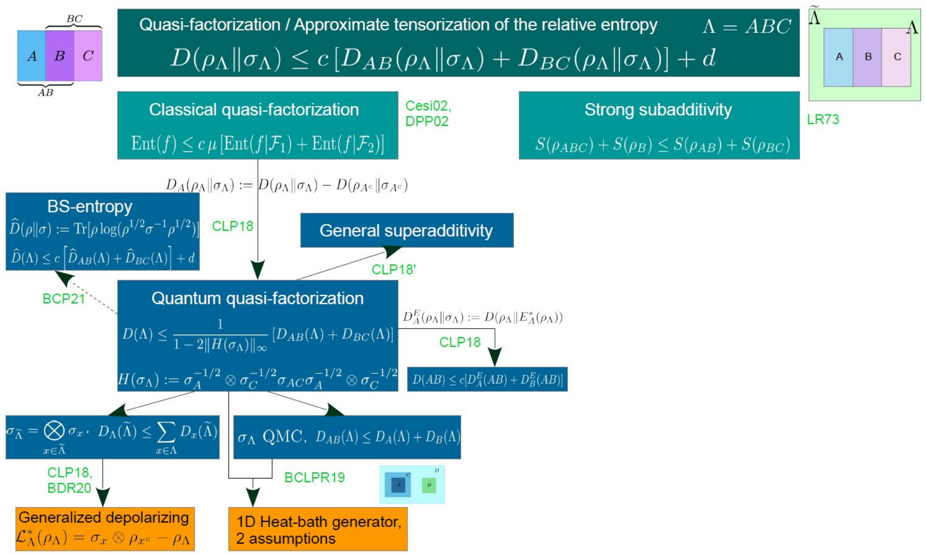




Quasi-factorization / Approximate tensorization



$$D_A^E(\rho_\Lambda\|\sigma_\Lambda) := D(\rho_\Lambda\|\sigma_\Lambda^{1/2}\sigma_{A^c}^{-1/2}\rho_{A^c}\sigma_{A^c}^{-1/2}\sigma_\Lambda^{1/2}).$$

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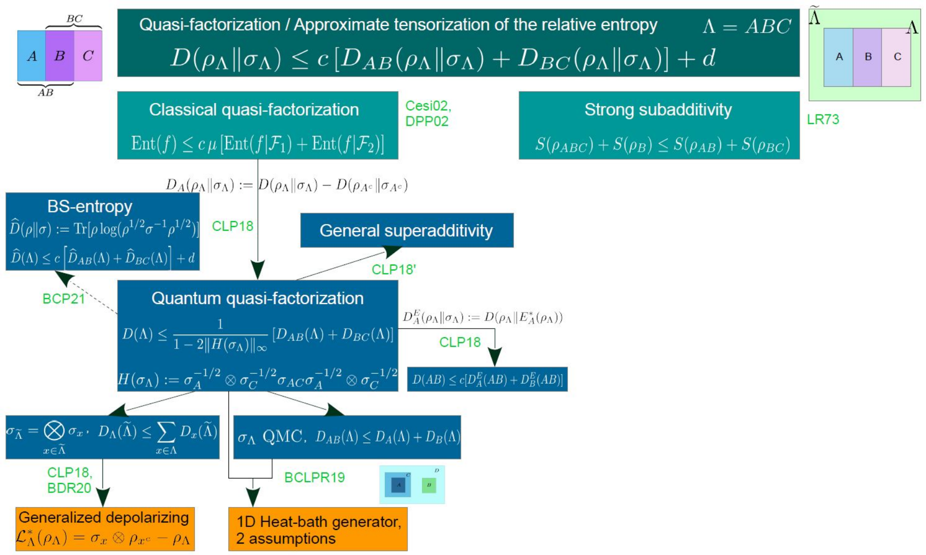

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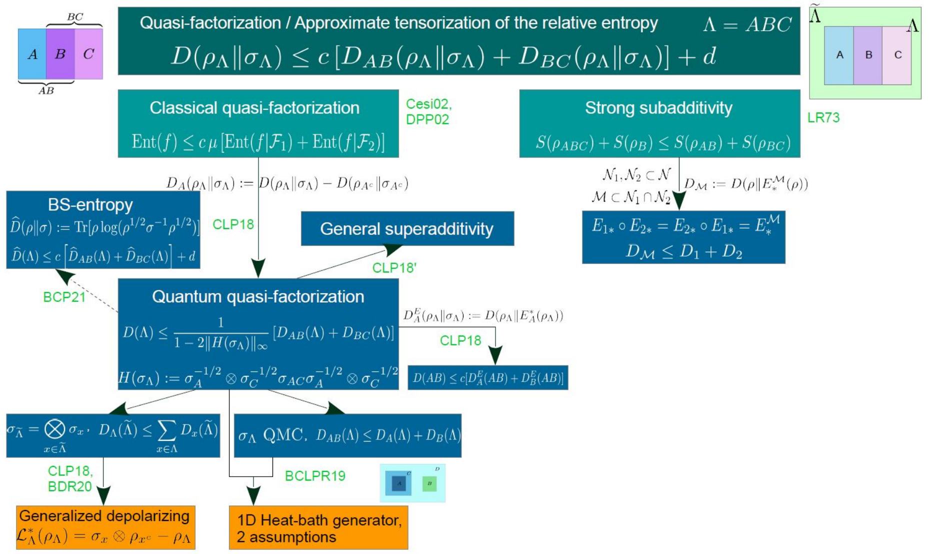

Generalization of strong subadditivity

In terms of the relative entropy, the **strong subadditivity of entropy** (Lieb-Ruskai '73) takes the form

$$D\left(\rho_{ABC} \middle\| \rho_B \otimes \frac{\mathbb{1}_{AC}}{d_{\mathcal{H}_{AC}}}\right) \leq D\left(\rho_{ABC} \middle\| \rho_{AB} \otimes \frac{\mathbb{1}_{C}}{d_{\mathcal{H}_{C}}}\right) + D\left(\rho_{ABC} \middle\| \rho_{BC} \otimes \frac{\mathbb{1}_{A}}{d_{\mathcal{H}_{A}}}\right) .$$

For $\mathcal{M} \subset \mathcal{N}_1, \mathcal{N}_2 \subset \mathcal{N}$, if $E^{\mathcal{M}}, E_1, E_2$ are the conditional expectations onto $\mathcal{M}, \mathcal{N}_1, \mathcal{N}_2$, respectively, we have

$$D(\rho \| E_*^{\mathcal{M}}(\rho)) \leq D(\rho \| E_{1*}(\rho)) + D(\rho \| E_{2*}(\rho)) \Leftrightarrow E_{1*} \circ E_{2*} = E_{2*} \circ E_{1*} = E_*^{\mathcal{M}} .$$

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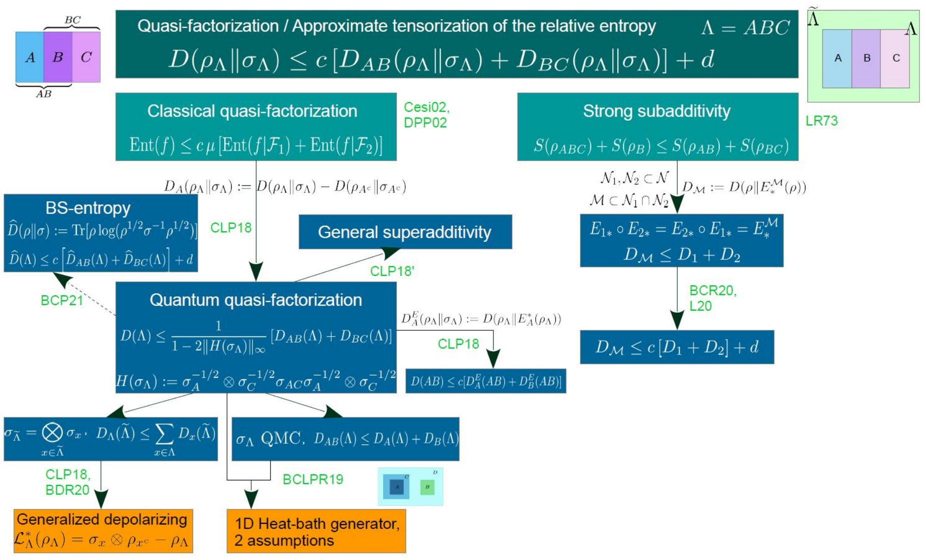

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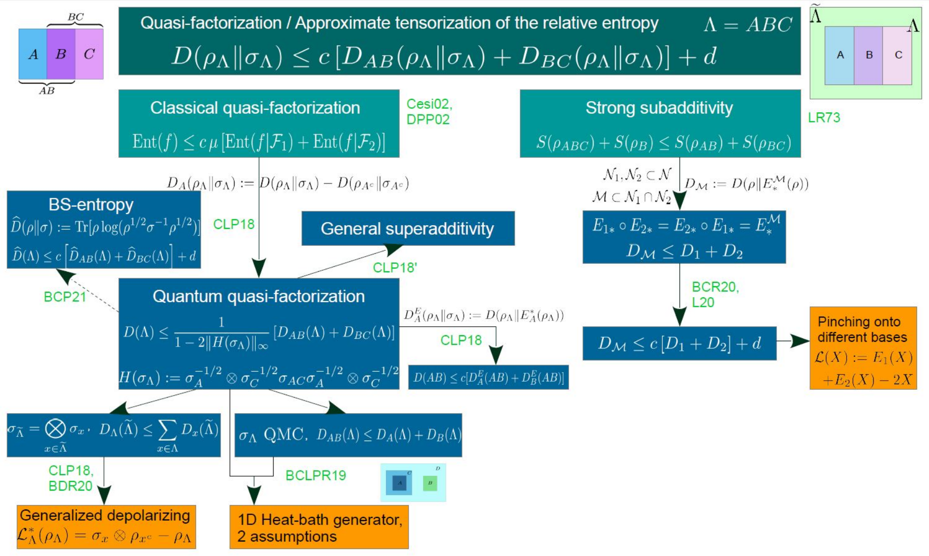

MLSI for Pinching onto different bases

$$\left\{ \left| e_k^{(1)} \right\rangle \right\} , \quad \left\{ \left| e_k^{(2)} \right\rangle \right\} \text{ orthonormal bases.}$$

$\mathcal{N}_1$, $\mathcal{N}_2$ diagonal onto first and second basis, respectively. $\mathcal{M} = \mathbb{C}\mathbb{1}_\ell$.

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and subsequently

$$\mathcal{L}(X) := E_1(X) + E_2(X) - 2X .$$

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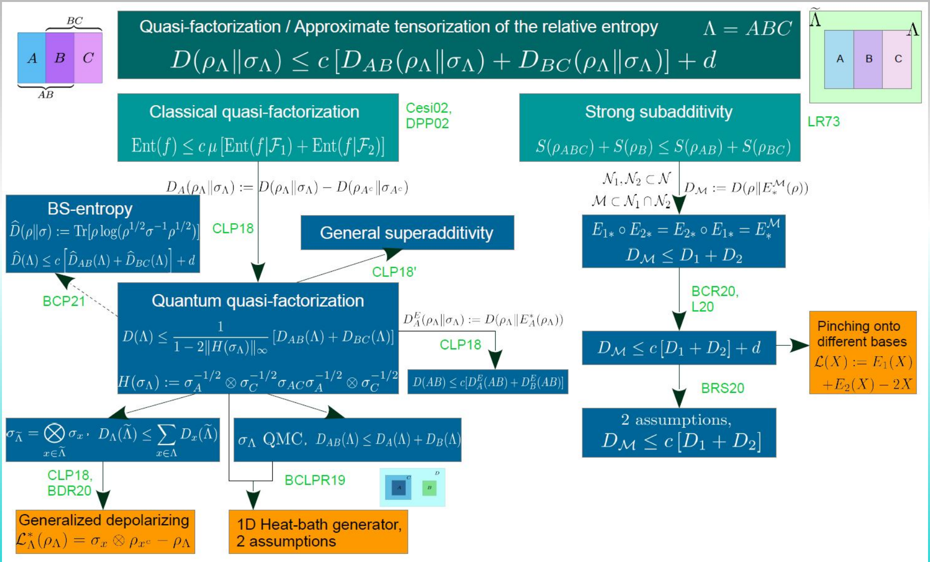

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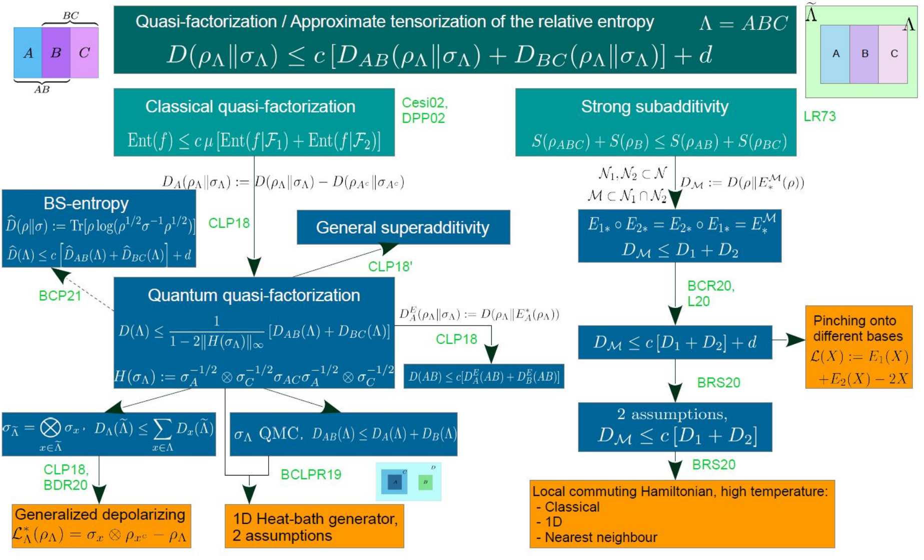

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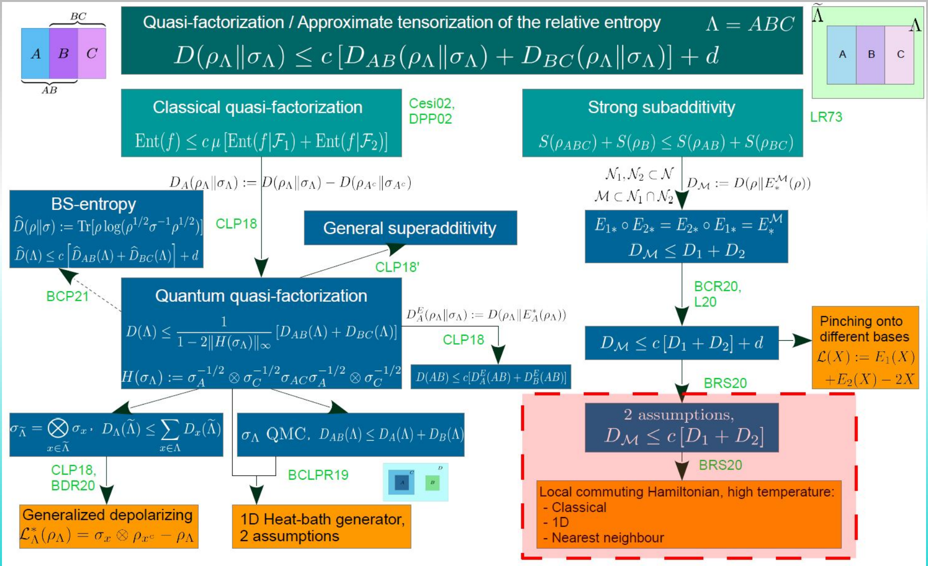
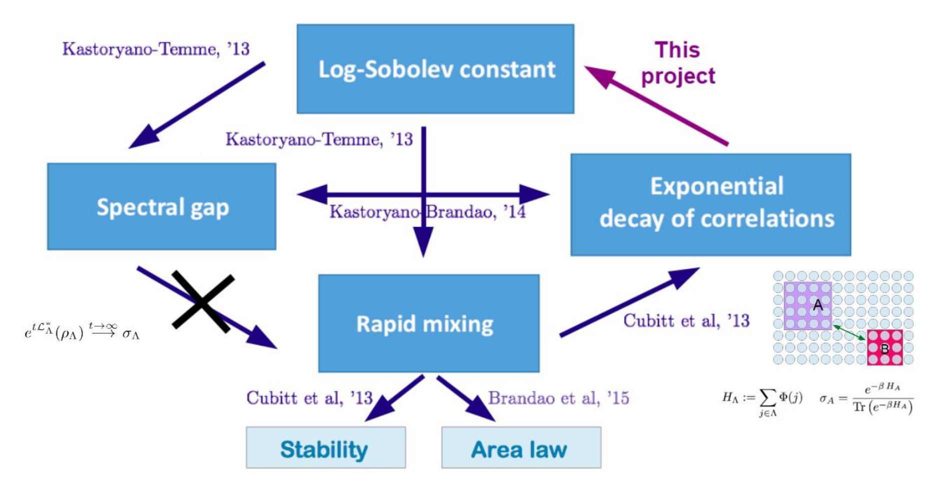


Quantum spin systems



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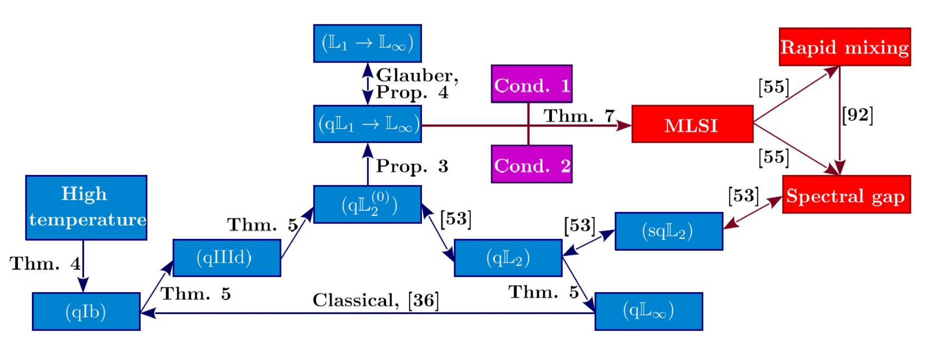

MLSI for quantum spin systems

MLSI, informal (C.-Rouzé-Stilck França '20)

Let H_Λ be a local commuting Hamiltonian such that one of the following conditions holds:

1. H_Λ is classical for $\beta < \beta_c$.
2. H_Λ is a nearest neighbour Hamiltonian for $\beta < \beta_c$.
3. Λ is 1D.

Then, there exists a local quantum Markov semigroup with fixed point σ_Λ, the Gibbs state of H_Λ, such that it has a positive **MLSI constant** which is independent of the system size.

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Modified logarithmic Sobolev inequality

Let $\left\{ \mathrm{e}^{t\mathcal{L}_\Lambda^*} \right\}_{t \geq 0}$ be a quantum Markov semigroup with $\mathcal{L}_\Lambda^*(\sigma_\Lambda) = 0$.

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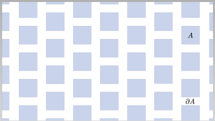
Tiling



Given a finite region $A \subset\subset \mathbb{Z}^d$, we decompose the fixed-point algebra $\mathcal{F}(\mathcal{L}_A)$ as

$$\mathcal{F}(\mathcal{L}_A) := \bigoplus_{i \in I_{\partial A}} \mathcal{B}(\mathcal{H}_i^A) \otimes \mathbb{1}_{\mathcal{K}_i^A} , \qquad \text{where} \qquad \mathcal{H}_\Lambda := \bigoplus_{i \in I_{\partial A}} \mathcal{H}_i^A \otimes \mathcal{K}_i^A .$$

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Condition 2

The covering $A = \bigcup_{i \in \mathcal{J}} A_i$ defined above satisfies:

(i) For all $i, j \in \mathcal{J}$, $E_{A_i} \circ E_{A_j} = E_{A_j} \circ E_{A_i} = E_{A_i \cup A_j}$; and

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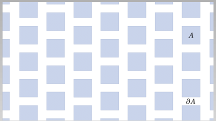

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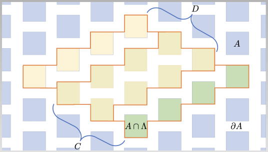
Clustering of correlations



Assuming frustration-freeness, for all $A \subset B \subset \Lambda \subset\subset \mathbb{Z}^d$, the blocks $P_i^B \mathcal{B}(\mathcal{H}_\Lambda) P_i^B$ are preserved by the conditional expectation E_A. Moreover, on each of these blocks, E_A only acts non-trivially on the factor $\mathcal{B}(\mathcal{K}_i^B)$, i.e. there exists a family of conditional expectations $\{E_A^{(i)} \in \mathcal{B}(\mathcal{B}(\mathcal{H}_{\mathcal{K}_i^B}))\}_{i \in I_{\partial B}}$ such that for each boundary condition $i \in I_{\partial B}$,

$$E_A|_{P_i^B \mathcal{B}(\mathcal{H}_\Lambda) P_i^B} := \mathrm{id}_{\mathcal{B}(\mathcal{H}_i^B)} \otimes E_A^{(i)} , \quad \text{with} \quad E_{A*}^{(i)}(\rho) := \sum_{j \in I_{\partial A}^i} \mathrm{tr}\left(P_j^{i,A} \, \rho \, P_j^{i,A} \right) \otimes \tau_j^{i,A} .$$

Clustering of correlations

$\mathcal{L}$ satisfies the $\mathbb{L}_1 \to \mathbb{L}_\infty$ clustering of correlations if there exist constants $c \geq 0$ and $\xi > 0$ such that for any intersecting $C, D \subset\subset \mathbb{Z}^d$,

$$\max_{i \in I_{\partial(C \cup D)}} \left\| E_C^{(i)} \circ E_D^{(i)} - E_{C \cup D}^{(i)} : \mathbb{L}_1(\tau_i^{C \cup D})_{\mathrm{sa}} \to \mathcal{B}(\mathcal{K}_i^{C \cup D})_{\mathrm{sa}} \right\| \leq c \, |C \cup D| \, \mathrm{e}^{-\frac{\mathrm{d}(C \setminus D, D \setminus C)}{\xi}} , \tag{q$\mathbb{L}_1 \to \mathbb{L}_\infty$}$$

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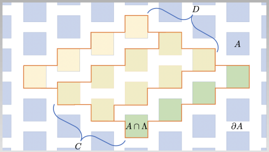

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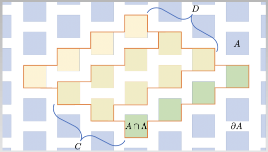
Approximate tensorization of the relative entropy



Approximate tensorization (C.-Rouzé-Stilck França '20)

Let $\mathcal{L}$ be a Gibbs sampler corresponding to a commuting potential. Assume further that the family $\mathcal{L}$ satisfies $\mathrm{q}\mathbb{L}_1 \to \mathbb{L}_\infty$ with parameters $c \geq 0$ and $\xi > 0$, as well as Condition 2. Then, for any $C, D \in \widetilde{\mathcal{S}}$ such that $C, D \subset \Lambda \subset\subset \mathbb{Z}^d$ with $2c\,|C \cup D|\exp\Big(-\frac{\mathrm{d}(C\setminus D, D\setminus C)}{\xi}\Big) < 1$, and all $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$,

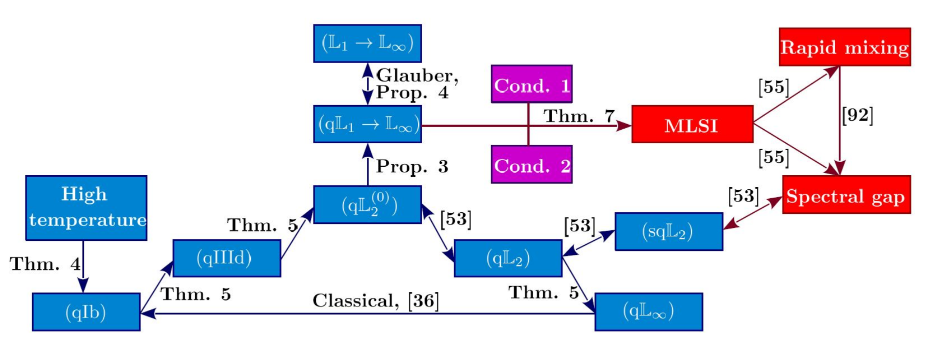
$$D(\omega \| E_{C\cup D *}(\omega)) \leq \frac{1}{1 - 2c\,|C\cup D|\,\mathrm{e}^{-\frac{\mathrm{d}(C\setminus D, D\setminus C)}{\xi}}} \Big(D(\omega\|E_{C*}(\omega)) + D(\omega\|E_{D*}(\omega)) \Big) ,$$

with $\omega := E_{A\cap\Lambda *}(\rho)$.

Here, we show that a condition on the **fixed points** of the generator and a condition of **decay of correlations** imply

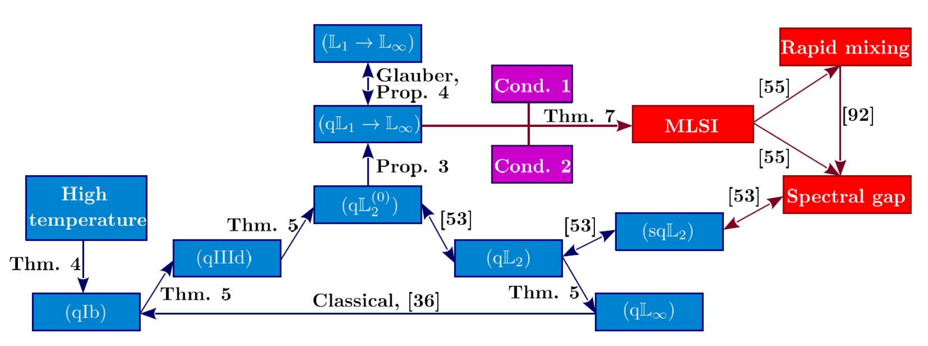
$$d = 0, \; c \sim 1 + \kappa\,\mathrm{e}^{-\,\mathrm{d}(C\setminus D, D\setminus C)} \; .$$

Quantum spin systems



$(q\mathbb{L}_1 \to \mathbb{L}_\infty)$ + Condition 2 $\Rightarrow$ Approximate tensorization

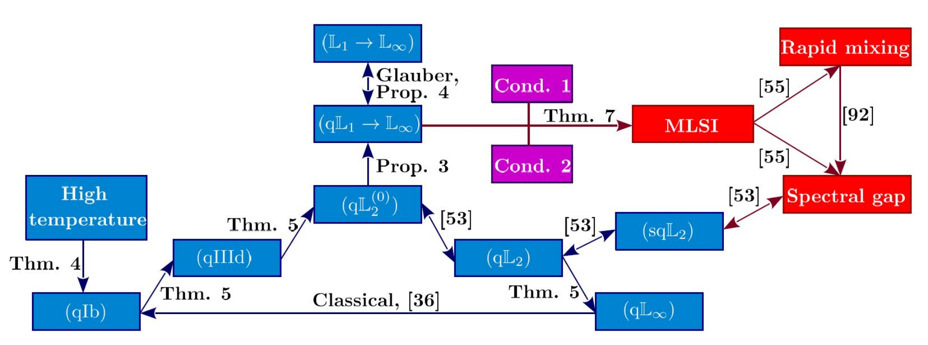
Quantum spin systems



$$(q\mathbb{L}_1 \to \mathbb{L}_\infty) + \text{Condition 2} \Rightarrow \text{Approximate tensorization}$$

Nearest neighbour Schmidt semigroups at high T satisfy both! (Bravyi-Vyalyi '05)

Quantum spin systems



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Condition 1 (Complete MLSI) + Approximate tensorization $\Rightarrow$ MLSI

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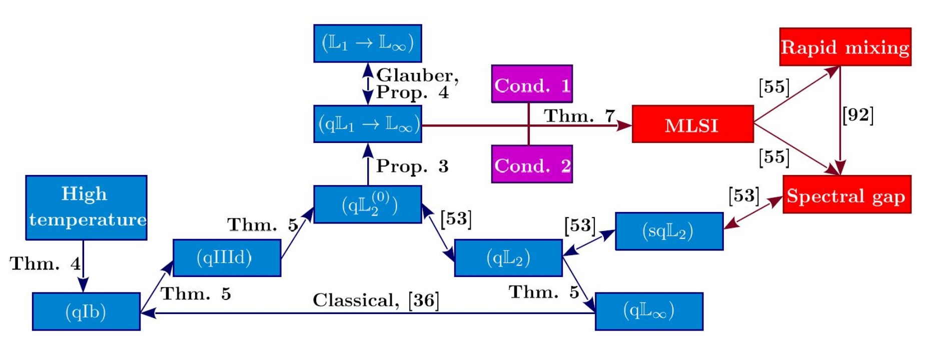

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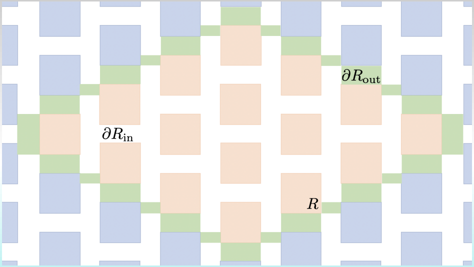
Peeling out

We want to show that there exists $\alpha > 0$, independent of the system size, such that

$$2\,\alpha D(\rho_\Lambda\|\sigma_\Lambda) \leq -\operatorname{tr}[\mathcal{L}^*_\Lambda(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)] =: \mathrm{EP}_\Lambda(\rho_\Lambda)\,.$$



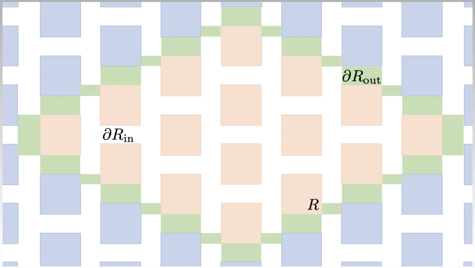
Chain rule for the relative entropy for any rhombus R:

$$D(\rho_\Lambda\|\sigma_\Lambda) = D(\rho_\Lambda\|E_{A\cap R*}(\rho_\Lambda)) + D(E_{A\cap R*}(\rho_\Lambda)\|\sigma_\Lambda)\ .$$

For $D(\rho_\Lambda\|E_{A\cap R*}(\rho_\Lambda))$, we use positivity of the complete MLSI (Junge-Gao-Laracuente '19, Rouzé-Gao '21)

$$\alpha_c := \inf_{k\in\mathbb{N}} \alpha\left(\mathcal{L}^*_\Lambda \otimes \mathbb{1}_k\right).$$

Peeling out



For $D(\rho_\Lambda \| E_C(\rho_\Lambda))$, we define a **pinched MLSI**

$$2\,\gamma_C\, D(E_{A*}(\rho_\Lambda)||E_{C*} \circ E_{A*}(\rho_\Lambda)) \leq -\operatorname{tr}[\mathcal{L}_C^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \quad .$$

and apply the approximate tensorization result on such quantity.

$$\begin{aligned} D(\rho_\Lambda \| E_{\Lambda*}(\rho_\Lambda)) &= D(\rho_\Lambda \| E_{A*}(\rho_\Lambda)) + D(E_{A*}(\rho_\Lambda) \| E_{\Lambda*}(\rho_\Lambda)) \\ &\leq \alpha_c(\mathcal{L}_{A*})^{-1} \mathrm{EP}_\Lambda(\rho_\Lambda) + \gamma_\Lambda^{-1} \mathrm{EP}_\Lambda(\rho_\Lambda) \\ &\leq \left(\alpha_c(\mathcal{L}_{A*})^{-1} + \gamma_\Lambda^{-1} \right) \mathrm{EP}_\Lambda(\rho_\Lambda) \end{aligned}$$

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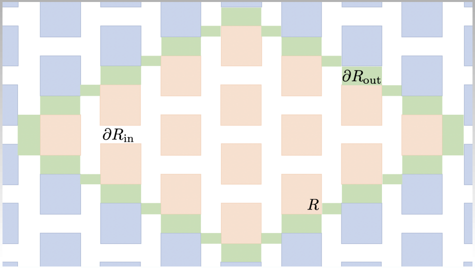

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Finally, we prove that γ_Λ^{-1} does not depend on $|\Lambda|$ (inspired by Cesi '02, Dai Pra-Paganoni-Posta '02).

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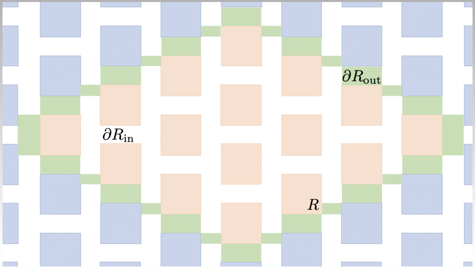

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THANK YOU FOR YOUR ATTENTION!