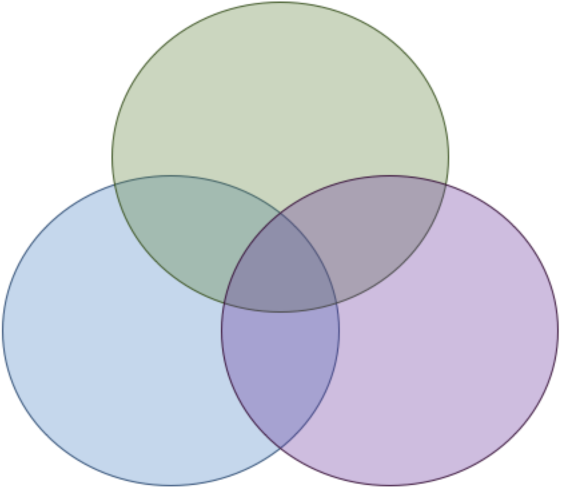
Quantum logarithmic Sobolev Inequalities for Quantum Many-Body Systems: An approach via Quasi-Factorization of the Relative Entropy

Ángela Capel Cuevas (ICMAT)

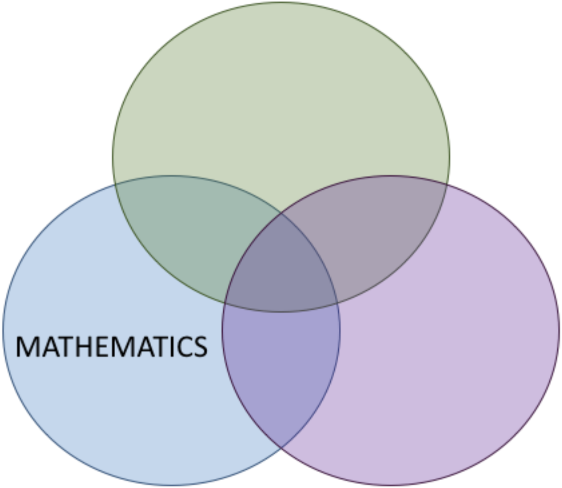
16 December 2019

Supervised by: David Pérez-García (UCM) and Angelo Lucia (Caltech)

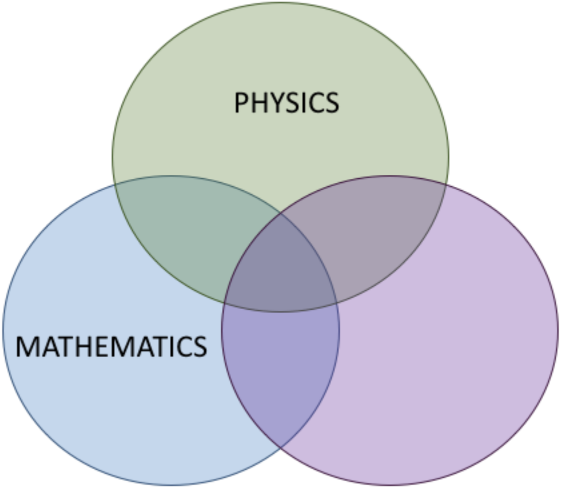
Field of study



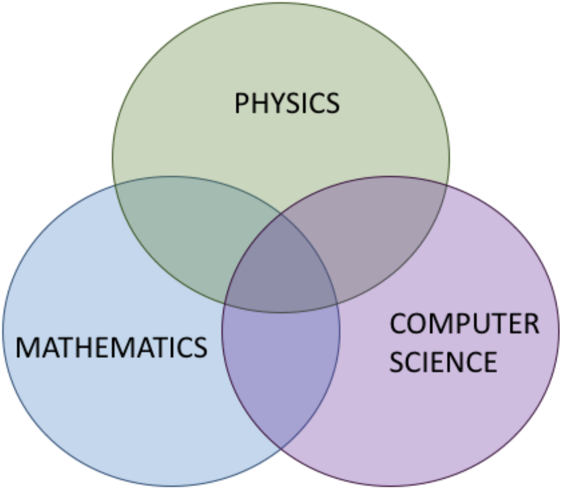
Field of study



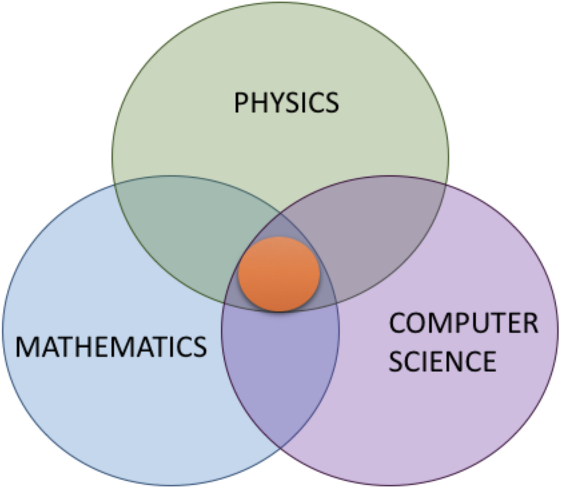
Field of study



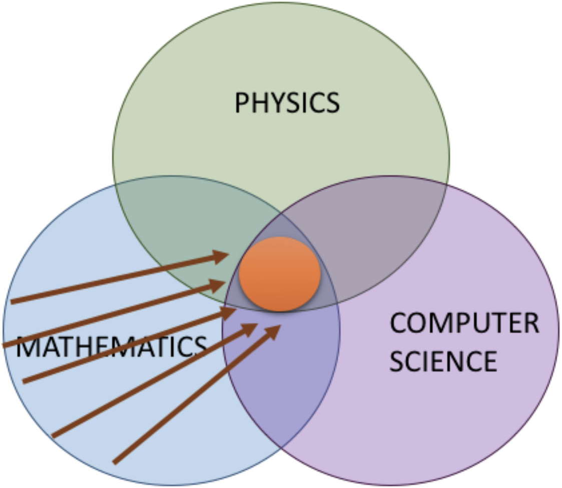
Field of study



Field of study



Field of study



Quantum

Q. information theory $\longleftrightarrow$ Q. many-body physics

Communication channels $\longleftrightarrow$ Physical interactions

Quantum

Q. information theory $\longleftrightarrow$ Q. many-body physics

Communication channels $\longleftrightarrow$ Physical interactions

Tools and ideas $\longrightarrow$ Solve problems

Quantum

Q. information theory $\longleftrightarrow$ Q. many-body physics

Communication channels $\longleftrightarrow$ Physical interactions

Tools and ideas $\longrightarrow$ Solve problems

Storage and transmision of information $\longleftarrow$ Models

Quantum

Q. information theory $\longleftrightarrow$ **Q. many-body physics**

Communication channels $\longleftrightarrow$ Physical interactions

Tools and ideas $\longrightarrow$ Solve problems

Storage and transmision of information $\longleftarrow$ Models

Main topic of this thesis

Field of study

Dissipative evolutions of quantum many-body systems

Main topic

Velocity of convergence of certain quantum dissipative evolutions to their thermal equilibriums.

Main topic of this thesis

Field of study

Dissipative evolutions of quantum many-body systems

Main topic

Velocity of convergence of certain quantum dissipative evolutions to their thermal equilibriums.

Concrete problem

Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant.

Main topic of this thesis

Field of study

Dissipative evolutions of quantum many-body systems

Main topic

Velocity of convergence of certain quantum dissipative evolutions to their thermal equilibriums.

Concrete problem

Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant.

Contents

1.1 Quantum dissipative systems

Open quantum systems

No experiment can be executed at zero temperature or be completely shielded from noise.

$\Rightarrow$ Open quantum many-body systems.

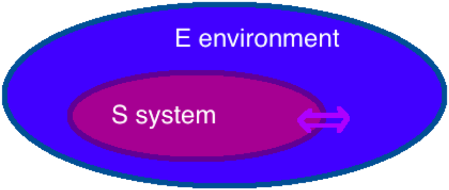


Figure: An open quantum many-body system.

Open quantum systems

No experiment can be executed at zero temperature or be completely shielded from noise.

$\Rightarrow$ Open quantum many-body systems.

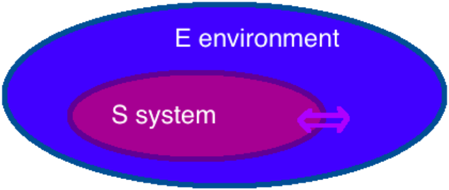


Figure: An open quantum many-body system.

- Dynamics of S is dissipative!

Open quantum systems

No experiment can be executed at zero temperature or be completely shielded from noise.

$\Rightarrow$ Open quantum many-body systems.

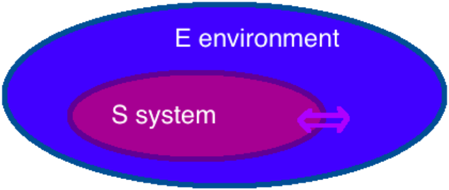
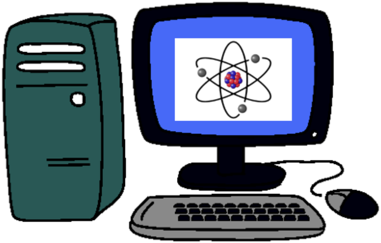


Figure: An open quantum many-body system.

- Dynamics of S is dissipative!

Quantum dissipative evolutions useful?

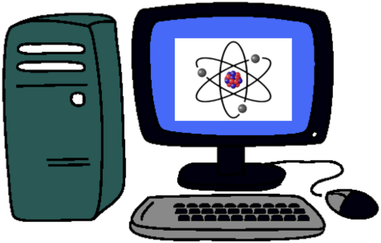
Main motivation:



One problem: Appearance of noise.

Quantum dissipative evolutions useful?

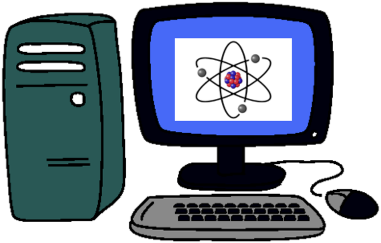
Main motivation:



One problem: Appearance of noise.
Some kinds of noise can be modelled using quantum dissipative evolutions.

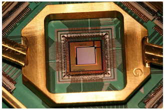
Quantum dissipative evolutions useful?

Main motivation:



One problem: Appearance of noise.
Some kinds of noise can be modelled using quantum dissipative evolutions.

Quantum dissipative evolutions useful?



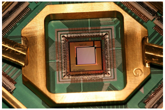
Recent change of perspective ⇒ Resource to exploit

New area:

Quantum dissipative engineering,

to create artificial evolutions in which the dissipative process works in favor (protecting the system from noisy evolutions).

Quantum dissipative evolutions useful?



Recent change of perspective ⇒ Resource to exploit

New area:

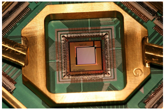
Quantum dissipative engineering,

to create artificial evolutions in which the dissipative process works in favor (protecting the system from noisy evolutions).

Interesting problems:

- Computational power
- Conditions against noise
- Time to obtain certain states
- ...

Quantum dissipative evolutions useful?



Recent change of perspective ⇒ Resource to exploit

New area:

Quantum dissipative engineering,

to create artificial evolutions in which the dissipative process works in favor (protecting the system from noisy evolutions).

Interesting problems:

- Computational power
- Conditions against noise
- Time to obtain certain states
- ...

Notation

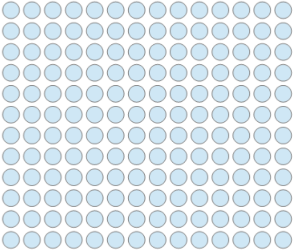


Figure: A quantum spin lattice system.

- Finite lattice $\Lambda \subset\subset \mathbb{Z}^d$.
- To every site $x \in \Lambda$ we associate $\mathcal{H}_x$ $(= \mathbb{C}^D)$.
- The global Hilbert space associated to Λ is $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$.
- The set of bounded linear endomorphisms on $\mathcal{H}_\Lambda$ is denoted by $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$.
- The set of density matrices is denoted by $\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{\rho_\Lambda \in \mathcal{B}_\Lambda \ : \ \rho_\Lambda \geq 0 \text{ and } \operatorname{tr}[\rho_\Lambda] = 1\}$.

Evolution of a system

Isolated system.

Physical evolution: $\rho \mapsto U\rho U^* \rightsquigarrow$ Reversible

Evolution of a system

Isolated system.

Physical evolution: $\rho \mapsto U\rho U^* \rightsquigarrow$ Reversible

Dissipative quantum system (non-reversible evolution)

$$\mathcal{T} : \rho \mapsto \mathcal{T}(\rho)$$

Evolution of a system

Isolated system.

Physical evolution: $\rho \mapsto U\rho U^* \rightsquigarrow$ Reversible

Dissipative quantum system (non-reversible evolution)

$$\mathcal{T} : \rho \mapsto \mathcal{T}(\rho)$$

- States to states $\Rightarrow$ Linear, positive and trace preserving.

Evolution of a system

Isolated system.

Physical evolution: $\rho \mapsto U\rho U^* \rightsquigarrow$ Reversible

Dissipative quantum system (non-reversible evolution)

$$\mathcal{T} : \rho \mapsto \mathcal{T}(\rho)$$

- States to states $\Rightarrow$ Linear, positive and trace preserving.

$\rho \otimes \sigma \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}')$, σ with trivial evolution

$$\left.\begin{array}{rcl} \hat{\mathcal{T}} : \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') & \rightarrow & \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') \\ \hat{\mathcal{T}}(\rho \otimes \sigma) & = & \mathcal{T}(\rho) \otimes \sigma \end{array}\right. \Rightarrow \hat{\mathcal{T}} = \mathcal{T} \otimes \mathbb{1}$$

Evolution of a system

Isolated system.

Physical evolution: $\rho \mapsto U\rho U^* \rightsquigarrow$ Reversible

Dissipative quantum system (non-reversible evolution)

$$\mathcal{T} : \rho \mapsto \mathcal{T}(\rho)$$

- States to states $\Rightarrow$ Linear, positive and trace preserving.

$$\rho \otimes \sigma \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}'),\ \sigma \text{ with trivial evolution}$$

$$\left.\begin{array}{rccc} \hat{\mathcal{T}} : & \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') & \rightarrow & \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') \\ & \hat{\mathcal{T}}(\rho \otimes \sigma) & = & \mathcal{T}(\rho) \otimes \sigma \end{array}\right. \Rightarrow \hat{\mathcal{T}} = \mathcal{T} \otimes \mathbb{1}$$

- Completely positive.

Evolution of a system

Isolated system.

Physical evolution: $\rho \mapsto U\rho U^* \rightsquigarrow$ Reversible

Dissipative quantum system (non-reversible evolution)

$$\mathcal{T} : \rho \mapsto \mathcal{T}(\rho)$$

- States to states $\Rightarrow$ Linear, positive and trace preserving.

$$\rho \otimes \sigma \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}'),\ \sigma \text{ with trivial evolution}$$

$$\begin{array}{rcl} \hat{\mathcal{T}} : \quad \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') & \rightarrow & \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') \\ \hat{\mathcal{T}}(\rho \otimes \sigma) & = & \mathcal{T}(\rho) \otimes \sigma \end{array} \Rightarrow \hat{\mathcal{T}} = \mathcal{T} \otimes \mathbb{1}$$

- Completely positive.

$\mathcal{T}$ quantum channel (CPTP map)

Evolution of a system

Isolated system.

Physical evolution: $\rho \mapsto U\rho U^* \rightsquigarrow$ Reversible

Dissipative quantum system (non-reversible evolution)

$$\mathcal{T} : \rho \mapsto \mathcal{T}(\rho)$$

- States to states $\Rightarrow$ Linear, positive and trace preserving.

$$\rho \otimes \sigma \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H}'), \ \sigma \text{ with trivial evolution}$$

$$\begin{array}{rccc} \hat{\mathcal{T}} : & \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') & \rightarrow & \mathcal{S}(\mathcal{H} \otimes \mathcal{H}') \\ & \hat{\mathcal{T}}(\rho \otimes \sigma) & = & \mathcal{T}(\rho) \otimes \sigma \end{array} \Rightarrow \hat{\mathcal{T}} = \mathcal{T} \otimes \mathbb{1}$$

- Completely positive.

$\mathcal{T}$ quantum channel (CPTP map)

Open systems

Open systems ⇒ Environment and system interact.

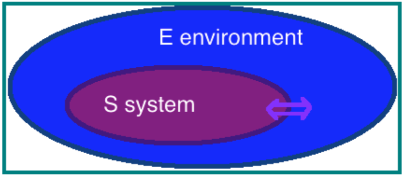


Figure: Environment + System form a closed system.

Open systems

Open systems $\Rightarrow$ Environment and system interact.

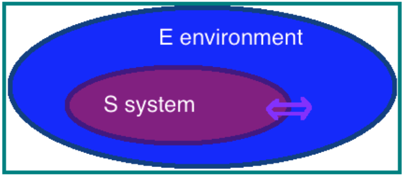


Figure: Environment + System form a closed system.

State for the environment: $|\psi\rangle\langle\psi|_E$

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|_E \mapsto U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^* \mapsto \mathrm{tr}_E[U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^*] = \tilde{\rho}$$

Open systems

Open systems $\Rightarrow$ Environment and system interact.

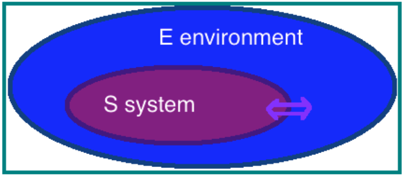


Figure: Environment + System form a closed system.

State for the environment: $|\psi\rangle\langle\psi|_E$

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|_E \mapsto U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^* \mapsto \operatorname{tr}_E[U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^*] = \tilde{\rho}$$

$$\begin{array}{rccc} \mathcal{T}: & \mathcal{S}(\mathcal{H}) & \rightarrow & \mathcal{S}(\mathcal{H}) \\ & \rho & \mapsto & \tilde{\rho} \end{array} \quad \text{quantum channel}$$

Open systems

Open systems $\Rightarrow$ Environment and system interact.

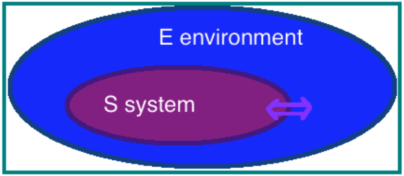


Figure: Environment + System form a closed system.

State for the environment: $|\psi\rangle\langle\psi|_E$

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|_E \mapsto U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^* \mapsto \mathrm{tr}_E[U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^*] = \tilde{\rho}$$

$$\begin{array}{cccc} \mathcal{T}: & \mathcal{S}(\mathcal{H}) & \rightarrow & \mathcal{S}(\mathcal{H}) \\ & \rho & \mapsto & \tilde{\rho} \end{array} \quad \text{quantum channel}$$

For every $t \geq 0$, the corresponding time slice is a realizable evolution $\mathcal{T}_t$ (quantum channel).

Open systems

Open systems $\Rightarrow$ Environment and system interact.

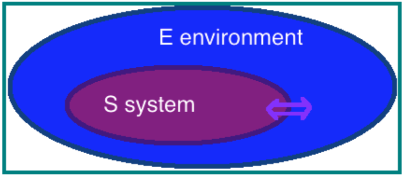


Figure: Environment + System form a closed system.

State for the environment: $|\psi\rangle\langle\psi|_E$

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|_E \mapsto U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^* \mapsto \operatorname{tr}_E[U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right)U^*] = \tilde{\rho}$$

$$\begin{array}{cccc} \mathcal{T}: & \mathcal{S}(\mathcal{H}) & \rightarrow & \mathcal{S}(\mathcal{H}) \\ & \rho & \mapsto & \tilde{\rho} \end{array} \quad \text{quantum channel}$$

For every $t \geq 0$, the corresponding time slice is a realizable evolution $\mathcal{T}_t$ (quantum channel).

Continuous-time description: Markovian approximation.

Open systems

Open systems $\Rightarrow$ Environment and system interact.

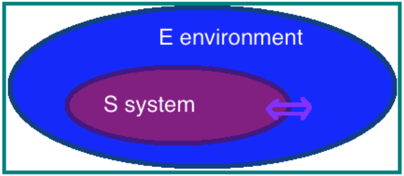


Figure: Environment + System form a closed system.

State for the environment: $|\psi\rangle\langle\psi|_E$

$$\rho \mapsto \rho \otimes |\psi\rangle\langle\psi|_E \mapsto U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right) U^* \mapsto \operatorname{tr}_E[U\left(\rho \otimes |\psi\rangle\langle\psi|_E\right) U^*] = \tilde{\rho}$$

$$\begin{array}{cccc} \mathcal{T}: & \mathcal{S}(\mathcal{H}) & \rightarrow & \mathcal{S}(\mathcal{H}) \\ & \rho & \mapsto & \tilde{\rho} \end{array} \quad \text{quantum channel}$$

For every $t \geq 0$, the corresponding time slice is a realizable evolution $\mathcal{T}_t$ (quantum channel).

Continuous-time description: Markovian approximation.

Dissipative quantum systems

Dissipative quantum systems

A **dissipative quantum system** is a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

Semigroup:

- $\mathcal{T}_t^* \circ \mathcal{T}_s^* = \mathcal{T}_{t+s}^*$.
- $\mathcal{T}_0^* = \mathbb{1}$.

Dissipative quantum systems

Dissipative quantum systems

A **dissipative quantum system** is a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

Semigroup:

- $\mathcal{T}_t^* \circ \mathcal{T}_s^* = \mathcal{T}_{t+s}^*$.
- $\mathcal{T}_0^* = \mathbb{1}$.

$$\frac{d}{dt}\mathcal{T}_t^* = \mathcal{T}_t^* \circ \mathcal{L}_\Lambda^* = \mathcal{L}_\Lambda^* \circ \mathcal{T}_t^*.$$

QMS generator

The infinitesimal generator $\mathcal{L}_\Lambda^*$ of the previous semigroup of quantum channels is usually called **Liouvillian**, or **Lindbladian**.

$$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*} \Leftrightarrow \mathcal{L}_\Lambda^* = \frac{d}{dt}\mathcal{T}_t^* \, |_{t=0}.$$

Dissipative quantum systems

Dissipative quantum systems

A **dissipative quantum system** is a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

Semigroup:

- $\mathcal{T}_t^* \circ \mathcal{T}_s^* = \mathcal{T}_{t+s}^*$.
- $\mathcal{T}_0^* = \mathbb{1}$.

$$\frac{d}{dt}\mathcal{T}_t^* = \mathcal{T}_t^* \circ \mathcal{L}_\Lambda^* = \mathcal{L}_\Lambda^* \circ \mathcal{T}_t^* .$$

QMS generator

The infinitesimal generator $\mathcal{L}_\Lambda^*$ of the previous semigroup of quantum channels is usually called **Liouvillian**, or **Lindbladian**.

$$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*} \Leftrightarrow \mathcal{L}_\Lambda^* = \frac{d}{dt}\mathcal{T}_t^* \mid_{t=0} .$$

Dissipative quantum systems

Primitive QMS

We assume that $\{\mathcal{T}_t^*\}_{t\geq 0}$ has a unique full-rank invariant state, which we denote by σ.

Reversibility

We also assume that the quantum Markov process studied is **reversible**, i.e., satisfies the **detailed balance condition**:

$$\langle f, \mathcal{L}(g)\rangle_\sigma = \langle \mathcal{L}(f), g\rangle_\sigma$$

for every $f, g \in \mathcal{A}$, in the Heisenberg picture.

Dissipative quantum systems

Primitive QMS

We assume that $\{\mathcal{T}_t^*\}_{t\geq 0}$ has a unique full-rank invariant state, which we denote by σ.

Reversibility

We also assume that the quantum Markov process studied is **reversible**, i.e., satisfies the **detailed balance condition**:

$$\langle f, \mathcal{L}(g)\rangle_\sigma = \langle \mathcal{L}(f), g\rangle_\sigma$$

for every $f, g \in \mathcal{A}$, in the Heisenberg picture.

Notation: $\rho_t := \mathcal{T}_t^*(\rho)$.

$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t^*(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$

Dissipative quantum systems

Primitive QMS

We assume that $\{\mathcal{T}_t^*\}_{t\geq 0}$ has a unique full-rank invariant state, which we denote by σ.

Reversibility

We also assume that the quantum Markov process studied is **reversible**, i.e., satisfies the **detailed balance condition**:

$$\langle f, \mathcal{L}(g)\rangle_\sigma = \langle \mathcal{L}(f), g\rangle_\sigma$$

for every $f, g \in \mathcal{A}$, in the Heisenberg picture.

Notation: $\rho_t := \mathcal{T}_t^*(\rho)$.

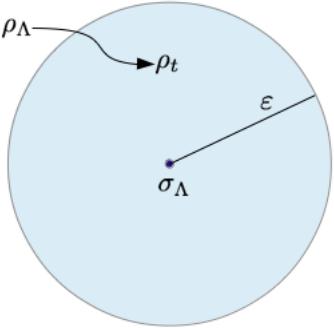
$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t^*(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$

Mixing time

Mixing time

We define the **mixing time** of $\{\mathcal{T}_t^*\}$ by

$$\tau(\varepsilon) = \min\left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \mathcal{T}_t^*(\rho) - \mathcal{T}_\infty^*(\rho) \right\|_1 \leq \varepsilon \right\}.$$

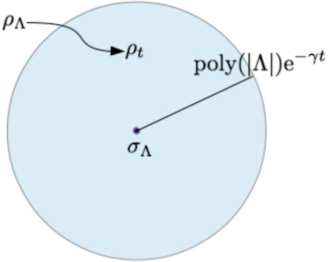


Rapid mixing

Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}.$$



Problem

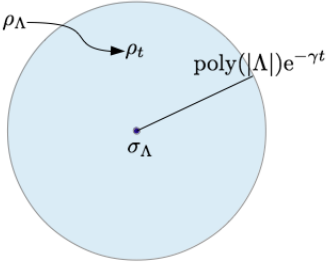
Find examples of rapid mixing!

Rapid mixing

Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}.$$

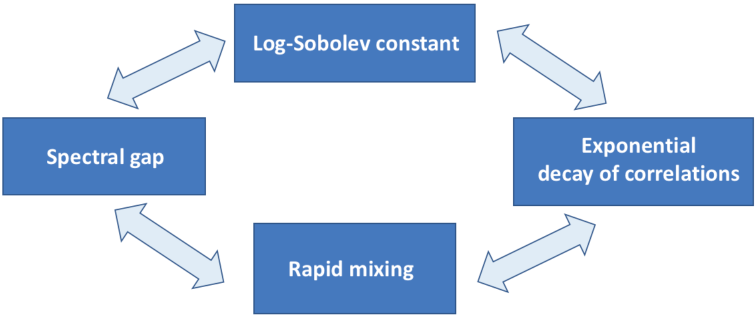


Problem

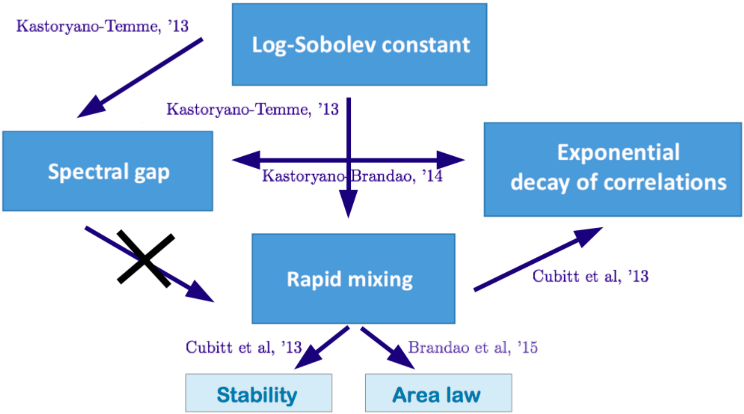
Find examples of rapid mixing!

1.2 Logarithmic Sobolev inequalities

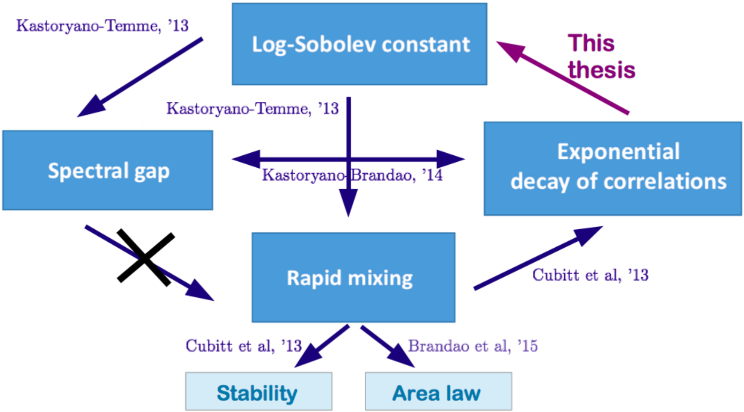
Classical spin systems



Quantum spin systems



Quantum spin systems



Log-Sobolev inequality (MLSI)

Recall: $\rho_t := \mathcal{T}_t^*(\rho)$.

Liouville's equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

Log-Sobolev inequality (MLSI)

Recall: $\rho_t := \mathcal{T}_t^*(\rho)$.

Liouville's equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

Relative entropy of ρ_t and σ_Λ:

$$D(\rho_t||\sigma_\Lambda) = \operatorname{tr}[\rho_t(\log \rho_t - \log \sigma_\Lambda)].$$

Log-Sobolev inequality (MLSI)

Recall: $\rho_t := \mathcal{T}_t^*(\rho)$.

Liouville's equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

Relative entropy of ρ_t and σ_Λ:

$$D(\rho_t || \sigma_\Lambda) = \operatorname{tr}[\rho_t(\log \rho_t - \log \sigma_\Lambda)].$$

Differentiating:

$$\partial_t D(\rho_t || \sigma_\Lambda) = \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_t)(\log \rho_t - \log \sigma_\Lambda)].$$

Log-Sobolev inequality (MLSI)

Recall: $\rho_t := \mathcal{T}_t^*(\rho)$.

Liouville's equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

Relative entropy of ρ_t and σ_Λ:

$$D(\rho_t || \sigma_\Lambda) = \operatorname{tr}[\rho_t(\log \rho_t - \log \sigma_\Lambda)].$$

Differentiating:

$$\partial_t D(\rho_t || \sigma_\Lambda) = \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_t)(\log \rho_t - \log \sigma_\Lambda)].$$

Lower bound for the derivative of $D(\rho_t||\sigma_\Lambda)$ in terms of itself:

$$2\alpha D(\rho_t || \sigma_\Lambda) \leq -\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_t)(\log \rho_t - \log \sigma_\Lambda)].$$

Log-Sobolev inequality (MLSI)

Recall: $\rho_t := \mathcal{T}_t^*(\rho)$.

Liouville's equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

Relative entropy of ρ_t and σ_Λ:

$$D(\rho_t||\sigma_\Lambda) = \operatorname{tr}[\rho_t(\log \rho_t - \log \sigma_\Lambda)].$$

Differentiating:

$$\partial_t D(\rho_t||\sigma_\Lambda) = \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_t)(\log \rho_t - \log \sigma_\Lambda)].$$

Lower bound for the derivative of $D(\rho_t||\sigma_\Lambda)$ in terms of itself:

$$2\alpha D(\rho_t||\sigma_\Lambda) \leq -\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_t)(\log \rho_t - \log \sigma_\Lambda)].$$

Log-Sobolev constant

Log-Sobolev constant

The **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ is defined as:

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

If $\liminf_{\Lambda \nearrow \mathbb{Z}^d} \alpha(\mathcal{L}_\Lambda^*) > 0$:

$$D(\rho_t || \sigma_\Lambda) \leq D(\rho_\Lambda || \sigma_\Lambda) e^{-2\,\alpha(\mathcal{L}_\Lambda^*)\,t},$$

Log-Sobolev constant

Log-Sobolev constant

The **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ is defined as:

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

If $\liminf_{\Lambda \nearrow \mathbb{Z}^d} \alpha(\mathcal{L}_\Lambda^*) > 0$:

$$D(\rho_t || \sigma_\Lambda) \leq D(\rho_\Lambda || \sigma_\Lambda) e^{-2\,\alpha(\mathcal{L}_\Lambda^*)\,t},$$

and with **Pinsker's inequality**, we have:

$$\left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \sqrt{2D(\rho_\Lambda || \sigma_\Lambda)}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t} \leq \sqrt{2\log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t}.$$

Log-Sobolev constant

Log-Sobolev constant

The **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ is defined as:

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

If $\liminf_{\Lambda \nearrow \mathbb{Z}^d} \alpha(\mathcal{L}_\Lambda^*) > 0$:

$$D(\rho_t || \sigma_\Lambda) \leq D(\rho_\Lambda || \sigma_\Lambda) e^{-2\,\alpha(\mathcal{L}_\Lambda^*)\,t},$$

and with **Pinsker's inequality**, we have:

$$\left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \sqrt{2D(\rho_\Lambda || \sigma_\Lambda)}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t} \leq \sqrt{2 \log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t}.$$

Using the spectral gap (Kastoryano-Temme '13):

$$\left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \sqrt{1/\sigma_{\min}}\, e^{-\lambda(\mathcal{L}_\Lambda^*)\,t}.$$

Log-Sobolev constant

Log-Sobolev constant

The **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ is defined as:

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

If $\liminf\limits_{\Lambda \nearrow \mathbb{Z}^d} \alpha(\mathcal{L}_\Lambda^*) > 0$:

$$D(\rho_t || \sigma_\Lambda) \leq D(\rho_\Lambda || \sigma_\Lambda) e^{-2\,\alpha(\mathcal{L}_\Lambda^*)\, t},$$

and with **Pinsker's inequality**, we have:

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2D(\rho_\Lambda || \sigma_\Lambda)}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\, t} \leq \sqrt{2 \log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\, t}.$$

Using the spectral gap (Kastoryano-Temme '13):

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{1/\sigma_{\min}}\, e^{-\lambda(\mathcal{L}_\Lambda^*)\, t}.$$

Log-Sobolev constant

Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}.$$

For thermal states, $\sigma_{\min} \sim \exp(|\Lambda|)$.

Log-Sobolev constant

Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}.$$

For thermal states, $\sigma_{\min} \sim \exp(|\Lambda|)$.

Log-Sobolev constant $\Rightarrow$ Rapid mixing.

Log-Sobolev constant

Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}.$$

For thermal states, $\sigma_{\min} \sim \exp(|\Lambda|)$.

Log-Sobolev constant $\Rightarrow$ Rapid mixing.

Problem

Find positive log-Sobolev constants!

Log-Sobolev constant

Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}.$$

For thermal states, $\sigma_{\min} \sim \exp(|\Lambda|)$.

Log-Sobolev constant $\Rightarrow$ Rapid mixing.

Problem

Find positive log-Sobolev constants!

First main objective of this thesis

Develop a strategy to find positive log Sobolev constants from static properties on the fixed point.

Second main objective of this thesis

Apply that strategy to certain dissipative dynamics.

First main objective of this thesis

Develop a strategy to find positive log Sobolev constants from static properties on the fixed point.

Second main objective of this thesis

Apply that strategy to certain dissipative dynamics.

2 Results

Based on:

1. **(Super)** A. Capel, A. Lucia and D. Pérez-García, **Superadditivity of Quantum Relative Entropy for General States**, *IEEE Trans. on Inf. Theory*, 64 (7) (2018), 4758–4765.

2. **(Q-Fact)** A. Capel, A. Lucia and D. Pérez-García, **Quantum Conditional Relative Entropy and Quasi-Factorization of the Relative Entropy**, *J. Phys. A: Math. Theor.*, 51 (2018), 484001.

3. **(BS-entropy)** A. Bluhm and A. Capel, **A strengthened data processing inequality for the Belavkin-Staszewski relative entropy**, *Rev. Math. Phys.*, to appear (2019).

4. **(Heat-bath)** I. Bardet, A. Capel, A. Lucia, D. Pérez-García and C. Rouzé, **On the modified logarithmic Sobolev inequality for the heat-bath dynamics for 1D systems**, preprint, arXiv: 1908.09004.

5. **(Davies)** I. Bardet, A. Capel and C. Rouzé, **Positivity of the modified logarithmic Sobolev constant for quantum Davies semigroups: the commuting case**, in preparation.

2.1 Strategy

Classical spin systems

(Cesi, Dai Pra-Paganoni-Posta, '02)

(1) Quasi-factorization of the entropy (in terms of a conditional entropy).

+

(2) Recursive geometric argument.
Lower bound for the global log-Sobolev constant in terms of the log-Sobolev constant of a size-fixed region.

Classical spin systems

(Cesi, Dai Pra-Paganoni-Posta, '02)

(1) Quasi-factorization of the entropy (in terms of a conditional entropy).

+

(2) Recursive geometric argument.
Lower bound for the global log-Sobolev constant in terms of the log-Sobolev constant of a size-fixed region.

+

(3) Decay of correlations on the Gibbs measure.

Classical spin systems

(Cesi, Dai Pra-Paganoni-Posta, '02)

(1) Quasi-factorization of the entropy (in terms of a conditional entropy).

$+$

(2) Recursive geometric argument.
Lower bound for the global log-Sobolev constant in terms of the log-Sobolev constant of a size-fixed region.

$+$

(3) Decay of correlations on the Gibbs measure.

$\Downarrow$

Positive log-Sobolev constant.

Classical spin systems

(Cesi, Dai Pra-Paganoni-Posta, '02)

(1) Quasi-factorization of the entropy (in terms of a conditional entropy).

$+$

(2) Recursive geometric argument.
Lower bound for the global log-Sobolev constant in terms of the log-Sobolev constant of a size-fixed region.

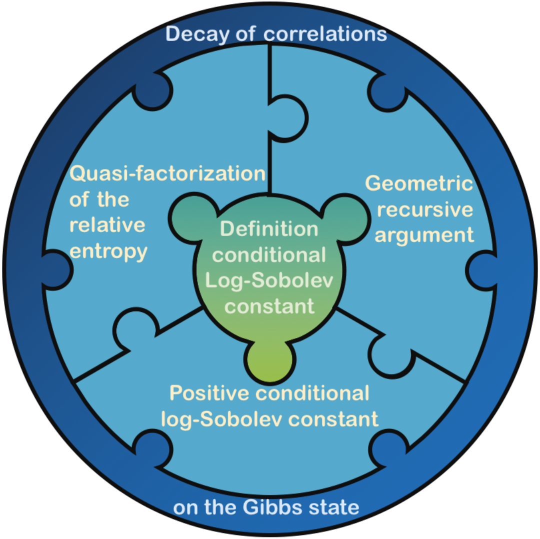
$+$

(3) Decay of correlations on the Gibbs measure.

$\Downarrow$

Positive log-Sobolev constant.

Quantum strategy



Conditional log-Sobolev constant

Log-Sobolev constant

Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ. We define the **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ by

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

Conditional log-Sobolev constant

Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian, with stationary state σ_Λ, such that $\mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*$, and $A \subseteq \Lambda$. We define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ on A by

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda || \sigma_\Lambda)}$$

Conditional log-Sobolev constant

Log-Sobolev constant

Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ. We define the **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ by

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

Conditional log-Sobolev constant

Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian, with stationary state σ_Λ, such that $\mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*$, and $A \subseteq \Lambda$. We define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ on A by

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_A(\rho_\Lambda || \sigma_\Lambda)}$$

Conditional relative entropy

Classical entropy and conditional entropy

Entropy:

$$\mathrm{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Conditional entropy:

$$\mathrm{Ent}_\mu(f \mid \mathcal{G}) = \mu(f(\log f - \log \mu(f \mid \mathcal{G})) \mid \mathcal{G}).$$

Quantum relative entropy

The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

$$D(\rho_\Lambda || \sigma_\Lambda) = \mathrm{tr}\left[\rho_\Lambda (\log \rho_\Lambda - \log \sigma_\Lambda)\right].$$

Conditional relative entropy

Classical entropy and conditional entropy

Entropy:

$$\mathrm{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Conditional entropy:

$$\mathrm{Ent}_\mu(f \mid \mathcal{G}) = \mu(f(\log f - \log \mu(f \mid \mathcal{G})) \mid \mathcal{G}).$$

Quantum relative entropy

The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

$$D(\rho_\Lambda || \sigma_\Lambda) = \mathrm{tr}\left[\rho_\Lambda (\log \rho_\Lambda - \log \sigma_\Lambda)\right].$$

Conditional relative entropy

Given a bipartite space $\mathcal{H}_{AB}$, we define the conditional relative entropy in A by:

$$D_A(\rho_{AB} || \sigma_{AB}) = D(\rho_{AB} || \sigma_{AB}) - D(\rho_B || \sigma_B)$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

(Q-Fact) $\rightarrow$ Axiomatic characterization of the CRE.

Conditional relative entropy

Classical entropy and conditional entropy

Entropy:

$$\mathrm{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Conditional entropy:

$$\mathrm{Ent}_\mu(f \mid \mathcal{G}) = \mu(f(\log f - \log \mu(f \mid \mathcal{G})) \mid \mathcal{G}).$$

Quantum relative entropy

The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

$$D(\rho_\Lambda || \sigma_\Lambda) = \mathrm{tr}\left[\rho_\Lambda (\log \rho_\Lambda - \log \sigma_\Lambda)\right].$$

Conditional relative entropy

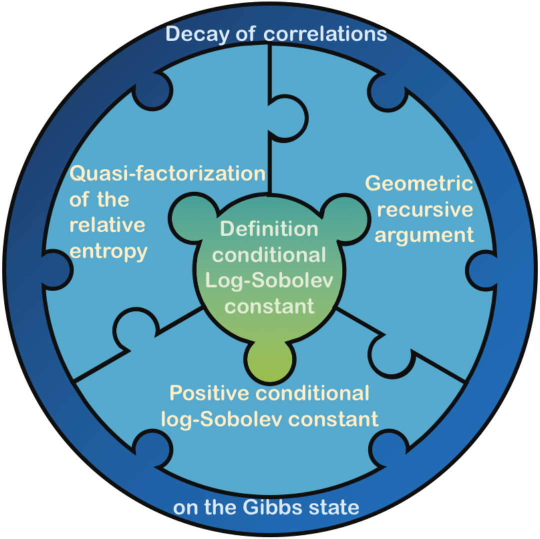
Given a bipartite space $\mathcal{H}_{AB}$, we define the conditional relative entropy in A by:

$$D_A(\rho_{AB} || \sigma_{AB}) = D(\rho_{AB} || \sigma_{AB}) - D(\rho_B || \sigma_B)$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

(Q-Fact) $\rightarrow$ Axiomatic characterization of the CRE.

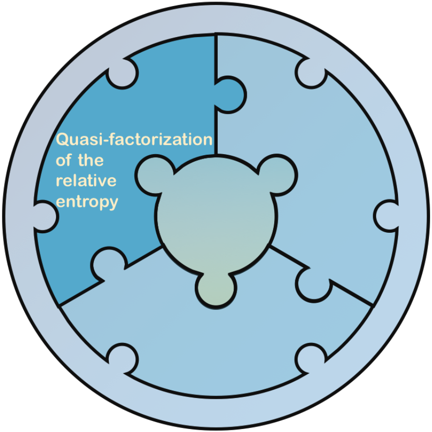
Quantum strategy



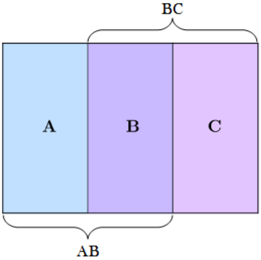
Based on:

1. **(Super)** A. Capel, A. Lucia and D. Pérez-García, **Superadditivity of Quantum Relative Entropy for General States**, *IEEE Trans. on Inf. Theory*, 64 (7) (2018), 4758–4765. **Quasi-Factorization**
2. **(Q-Fact)** A. Capel, A. Lucia and D. Pérez-García, **Quantum Conditional Relative Entropy and Quasi-Factorization of the Relative Entropy**, *J. Phys. A: Math. Theor.*, 51 (2018), 484001. **Quasi-Factorization**
3. **(BS-entropy)** A. Bluhm and A. Capel, **A strengthened data processing inequality for the Belavkin-Staszewski relative entropy**, *Rev. Math. Phys.*, to appear (2019).
4. **(Heat-bath)** I. Bardet, A. Capel, A. Lucia, D. Pérez-García and C. Rouzé, **On the modified logarithmic Sobolev inequality for the heat-bath dynamics for 1D systems**, preprint, arXiv: 1908.09004. **Log-Sobolev**
5. **(Davies)** I. Bardet, A. Capel and C. Rouzé, **Positivity of the modified logarithmic Sobolev constant for quantum Davies semigroups: the commuting case**, in preparation. **Log-Sobolev**

2.2 Part 2: Quasi-factorization of the relative entropy



Statement of the problem



Problem (Quasi-factorization of the relative entropy)

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in S_{ABC}$. Can we prove something like

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right]$$

where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$?

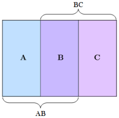


Figure: Choice of indices in $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Result of **quasi-factorization** of the relative entropy, for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$:

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right].$$

Quasi-factorization for the CRE, **(Q-Fact)**

In the previous inequality,

$$\xi(\sigma_{ABC}) = \frac{1}{1 - 2\| H(\sigma_{AC}) \|_\infty},$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \, \sigma_{AC} \, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Note that $H(\sigma_{AC}) = 0$ if σ_{AC} is a tensor product between A and C.

$$
\begin{aligned}
(1-2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) &\leq \\
& D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) = \\
& = 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).
\end{aligned}
$$

$$\Leftrightarrow$$

$$(1+2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

$$\begin{aligned}(1 - 2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \leq \\ D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) = \\ = 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).\end{aligned}$$

$$\Leftrightarrow$$

$$(1 + 2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

$$\Leftrightarrow$$

$$(1 + 2\|H(\sigma_{AC})\|_\infty) D(\rho_{AC}||\sigma_{AC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

$$
\begin{aligned}
(1 - 2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \leq & \\
D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) &= \\
&= 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).
\end{aligned}
$$

$$\Leftrightarrow$$

$$(1 + 2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

$$\Leftrightarrow$$

$$(1 + 2\|H(\sigma_{AC})\|_\infty) D(\rho_{AC}||\sigma_{AC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

This result is equivalent to **(Super)**:

$$\boxed{(1+2\|H(\sigma_{AB})\|_{\infty})D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

This result is equivalent to **(Super)**:

$$\boxed{(1 + 2\|H(\sigma_{AB})\|_\infty) D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

Due to:

- **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

we have

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

This result is equivalent to **(Super)**:

$$\boxed{(1+2\|H(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

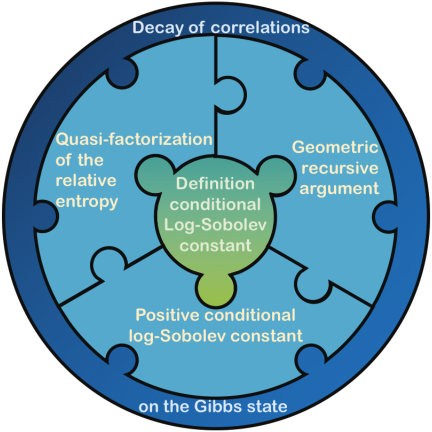
Due to:

- **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

we have

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

2.3 Part 3: Log-Sobolev constants



Example 1 (Q-Fact)

Heat-bath dynamics with tensor product fixed point

Heat-bath with tensor product fixed point

Theorem (Q-Fact)

The **heat-bath dynamics**, with tensor product fixed point, has a positive log-Sobolev constant.

Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x\in\Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x\in\Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

Heat-bath with tensor product fixed point

Theorem (Q-Fact)

The **heat-bath dynamics**, with tensor product fixed point, has a positive log-Sobolev constant.

Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x \in \Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

General depolarizing semigroup

Heat-bath with tensor product fixed point

Theorem (Q-Fact)

The **heat-bath dynamics**, with tensor product fixed point, has a positive log-Sobolev constant.

Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*$$

Since

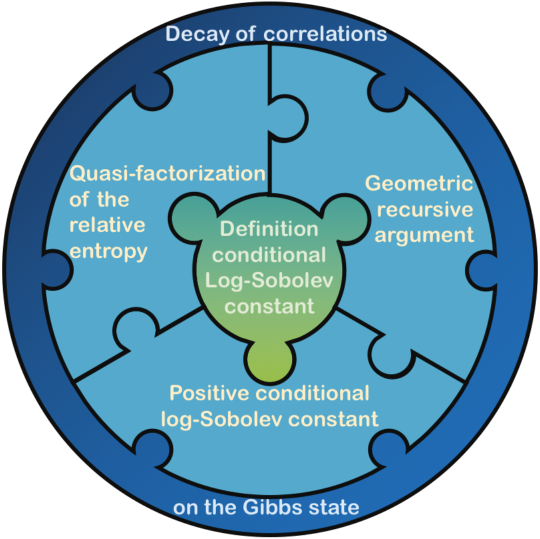
$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

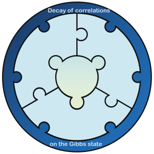
$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x \in \Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

General depolarizing semigroup

Strategy


Decay of correlations
Quasi-factorization of the relative entropy
Geometric recursive argument
Definition conditional Log-Sobolev constant
Positive conditional log-Sobolev constant
on the Gibbs state

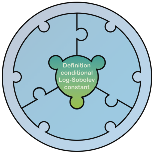
Heat-bath with tensor product fixed point



Assumption

$$\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x .$$

Heat-bath with tensor product fixed point



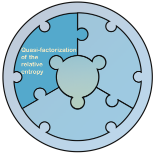
Conditional log-Sobolev constant

For $x \in \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in x by

$$\alpha_\Lambda(\mathcal{L}_x^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_x(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and $D_x(\rho_\Lambda || \sigma_\Lambda)$ is the conditional relative entropy.

Heat-bath with tensor product fixed point

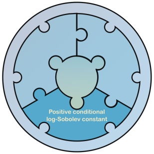


General quasi-factorization for σ a tensor product

Let $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ and $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x\in\Lambda} \sigma_x$. The following inequality holds:

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x\in\Lambda} D_x(\rho_\Lambda || \sigma_\Lambda).$$

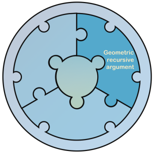
Heat-bath with tensor product fixed point



Lemma (Positivity of the conditional log-Sobolev constant)

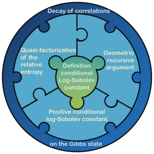
$$\alpha_\Lambda(\mathcal{L}_x^*) \geq \frac{1}{2}.$$

Heat-bath with tensor product fixed point



$$
\begin{aligned}
D(\rho_\Lambda || \sigma_\Lambda) &\leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \\
&\leq \sum_{x \in \Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2 \alpha_\Lambda(\mathcal{L}_x^*)} \\
&\leq \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x \in \Lambda} -\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\
&= \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \left(-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]\right) \\
&\leq \left(-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]\right) .
\end{aligned}
$$

Heat-bath with tensor product fixed point



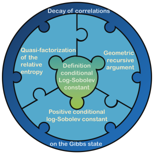
Positive log-Sobolev constant

$$\alpha(\mathcal{L}_\Lambda^*) \geq \frac{1}{2}.$$

Previous results:

- **Müller-Hermes et al. '15.** Lower bound 1/2 for the usual depolarizing semigroup, with fixed point $\mathbb{1}/d$.
- **Temme et al. '14.** For this semigroup, the log-Sobolev constant is positive, with a lower bound that is not universal.

Heat-bath with tensor product fixed point



Positive log-Sobolev constant

$$\alpha(\mathcal{L}_\Lambda^*) \geq \frac{1}{2}.$$

Previous results:

- **Müller-Hermes et al. '15.** Lower bound $1/2$ for the usual depolarizing semigroup, with fixed point $\mathbb{1}/d$.
- **Temme et al. '14.** For this semigroup, the log-Sobolev constant is positive, with a lower bound that is not universal.

Example 2, (Heat-bath)

Heat-bath dynamics in 1D

Heat-bath dynamics in 1D

$\sigma_\Lambda = \frac{\mathrm{e}^{-\beta H}}{\mathrm{tr}\left(\mathrm{e}^{-\beta H}\right)}$ is the Gibbs state of a k-local, commuting Hamiltonian H.

Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*,$$

with

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2},$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$,

Heat-bath dynamics in 1D

$\sigma_\Lambda = \frac{e^{-\beta H}}{\operatorname{tr}\left(e^{-\beta H}\right)}$ is the Gibbs state of a k-local, commuting Hamiltonian H.

Consider the local and global Lindbladians

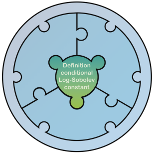
$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*,$$

with

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2},$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$,

Heat-bath dynamics in 1D



Conditional log-Sobolev constant

For $A \subset \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in A by

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2 D_A(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and

$$D_A(\rho_\Lambda || \sigma_\Lambda) = D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{A^c} || \sigma_{A^c}).$$

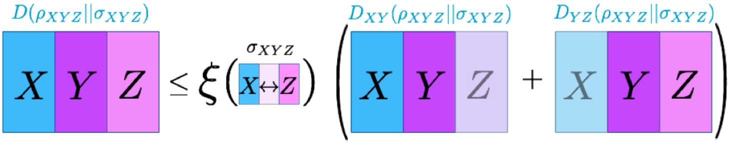
Quasi-factorization for the CRE (Q-Fact)

Let $\mathcal{H}_{XYZ}$ and $\rho_{XYZ}, \sigma_{XYZ} \in \mathcal{S}_{XYZ}$. The following holds

$$D(\rho_{XYZ}||\sigma_{XYZ}) \leq \xi(\sigma_{XZ}) \left[D_{XY}(\rho_{XYZ}||\sigma_{XYZ}) + D_{YZ}(\rho_{XYZ}||\sigma_{XYZ}) \right],$$

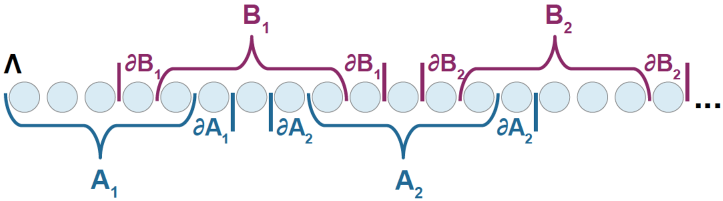
where

$$\xi(\sigma_{XZ}) = \frac{1}{1 - 2\left\| \sigma_X^{-1/2} \otimes \sigma_Z^{-1/2} \, \sigma_{XZ} \, \sigma_X^{-1/2} \otimes \sigma_Z^{-1/2} - \mathbb{1}_{XZ} \right\|_\infty}.$$



Quasi-factorization of the relative entropy

STEP 1



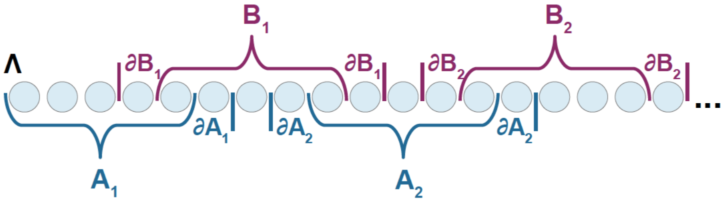
$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{j=1}^{n} B_j$$

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2 \| h(\sigma_{A^c B^c}) \|_\infty} \left[D_A(\rho_\Lambda || \sigma_\Lambda) + D_B(\rho_\Lambda || \sigma_\Lambda) \right],$$

$$h(\sigma_{A^c B^c}) := \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^c B^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - \mathbb{1}_{A^c B^c}.$$

Quasi-factorization of the relative entropy

STEP 1



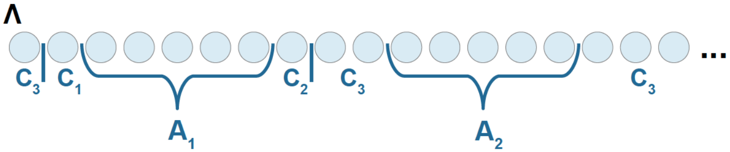
$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{j=1}^{n} B_j$$

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2\| h(\sigma_{A^c B^c}) \|_\infty} \left[D_A(\rho_\Lambda || \sigma_\Lambda) + D_B(\rho_\Lambda || \sigma_\Lambda) \right],$$

$$h(\sigma_{A^c B^c}) := \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^c B^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - \mathbb{1}_{A^c B^c}.$$

Sketch of the proof

STEP 2



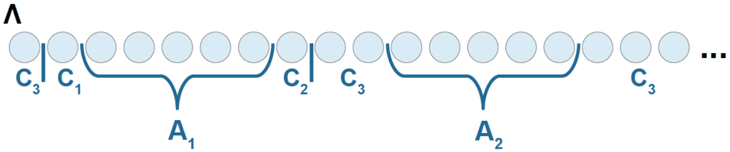
$$D_A(\rho_\Lambda||\sigma_\Lambda) \leq \sum_{i=1}^{n} D_{A_i}(\rho_\Lambda||\sigma_\Lambda)$$

σ_Λ is a QMC between $A_1 \leftrightarrow \partial A_1 \leftrightarrow \Lambda \setminus (A_1 \cup \partial A_1)$

$$\sigma_\Lambda = \bigoplus_{i \in I} \sigma_{A_1 (\partial a_1)_i^L} \otimes \sigma_{(\partial a_1)_i^R \Lambda \setminus (A_1 \cup \partial A_1)}$$

Sketch of the proof

STEP 2

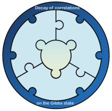


$$D_A(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{i=1}^{n} D_{A_i}(\rho_\Lambda || \sigma_\Lambda)$$

σ_Λ is a QMC between $A_1 \leftrightarrow \partial A_1 \leftrightarrow \Lambda \setminus (A_1 \cup \partial A_1)$

$$\sigma_\Lambda = \bigoplus_{i \in I} \sigma_{A_1 (\partial a_1)_i^L} \otimes \sigma_{(\partial a_1)_i^R \Lambda \setminus (A_1 \cup \partial A_1)}$$

Heat-bath dynamics in 1D



Assumption 1

In a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$, A and B not connected, we have

$$\left\| h(\sigma_{AB}) \right\|_\infty = \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty \leq K < \frac{1}{2}.$$

In particular, Gibbs states at high-enough temperature satisfy this.

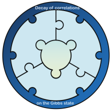
Assumption 2

For any $B \subset \Lambda$, $B = B_1 \cup B_2$, it holds:

$$D_B(\rho_\Lambda || \sigma_\Lambda) \leq f(\sigma_{B\partial}) \left(D_{B_1}(\rho_\Lambda || \sigma_\Lambda) + D_{B_2}(\rho_\Lambda || \sigma_\Lambda) \right).$$

In particular, tensor products satisfy this (with $f = 1$).

Heat-bath dynamics in 1D



Assumption 1

In a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$, A and B not connected, we have

$$\left\| h(\sigma_{AB}) \right\|_\infty = \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty \leq K < \frac{1}{2}.$$

In particular, Gibbs states at high-enough temperature satisfy this.

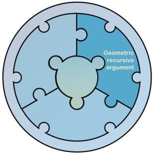
Assumption 2

For any $B \subset \Lambda$, $B = B_1 \cup B_2$, it holds:

$$D_B(\rho_\Lambda || \sigma_\Lambda) \leq f(\sigma_{B\partial}) \left(D_{B_1}(\rho_\Lambda || \sigma_\Lambda) + D_{B_2}(\rho_\Lambda || \sigma_\Lambda) \right).$$

In particular, tensor products satisfy this (with $f = 1$).

Heat-bath dynamics in 1D



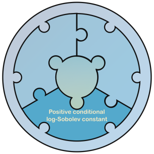
STEP 3

$$\text{Assumption 1} \Rightarrow \alpha(\mathcal{L}_\Lambda^*) \geq \tilde{K} \min_{i \in \{1, \ldots n\}} \left\{ \alpha_\Lambda(\mathcal{L}_{A_i}^*), \alpha_\Lambda(\mathcal{L}_{B_i}^*) \right\}$$

Using locality of the Lindbladian

$$\mathcal{L}_A^* + \mathcal{L}_B^* = \mathcal{L}_{A \cup B}^* + \mathcal{L}_{A \cap B}^* .$$

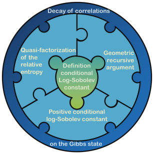
Heat-bath dynamics in 1D



STEP 4

$$\text{Assumption } 2 \Rightarrow \alpha_\Lambda(\mathcal{L}^*_{A_i}) \geq g(\sigma_{A_i\partial}) > 0.$$

Heat-bath dynamics in 1D



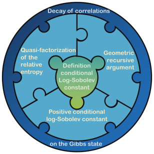
Theorem (Heat-bath)

In 1D, if Assumptions 1 and 2 hold, for a k-local commuting Hamiltonian, the heat-bath dynamics has a positive log-Sobolev constant.

Previous results:

- **Kastoryano-Brandao, '15.** In 1D, for a k-local commuting Hamiltonian, the heat-bath dynamics is always gapped.

Heat-bath dynamics in 1D



Theorem (Heat-bath)

In 1D, if Assumptions 1 and 2 hold, for a k-local commuting Hamiltonian, the heat-bath dynamics has a positive log-Sobolev constant.

Previous results:

- **Kastoryano-Brandao, '15.** In 1D, for a k-local commuting Hamiltonian, the heat-bath dynamics is always gapped.

Example 3 (Davies)

Davies dynamics

Davies dynamics

Generator

The generator of the Davies dynamics is of the following form:

$$\mathcal{L}_\Lambda^\beta(X) = i[H_\Lambda, X] + \sum_{k\in\Lambda} \mathcal{L}_k^\beta(X) \, ,$$

where

$$\mathcal{L}_k^\beta(X) = \sum_{\omega,\alpha} \chi_{\alpha,k}^\beta(\omega) \left(S_{\alpha,k}^*(\omega) X S_{\alpha,k}(\omega) - \frac{1}{2} \left\{ S_{\alpha,k}^*(\omega) S_{\alpha,k}(\omega), X \right\} \right) .$$

Important property: Given $A \subseteq \Lambda$,

$$\mathcal{E}_A^\beta(X) := \mathcal{E}(X|\mathcal{N}_A) = \lim_{t\to\infty} \mathrm{e}^{t\mathcal{L}_A^\beta}(X) \, .$$

is a conditional expectation onto the subalgebra of fixed points of $\mathcal{L}_A^\beta$.

Davies dynamics

Generator

The generator of the Davies dynamics is of the following form:

$$\mathcal{L}_\Lambda^\beta(X) = i[H_\Lambda, X] + \sum_{k\in\Lambda} \mathcal{L}_k^\beta(X) \, ,$$

where

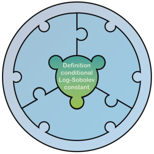
$$\mathcal{L}_k^\beta(X) = \sum_{\omega,\alpha} \chi_{\alpha,k}^\beta(\omega) \left(S_{\alpha,k}^*(\omega) X S_{\alpha,k}(\omega) - \frac{1}{2} \left\{ S_{\alpha,k}^*(\omega) S_{\alpha,k}(\omega), X \right\} \right) .$$

Important property: Given $A \subseteq \Lambda$,

$$\mathcal{E}_A^\beta(X) := \mathcal{E}(X|\mathcal{N}_A) = \lim_{t\to\infty} \mathrm{e}^{t\mathcal{L}_A^\beta}(X) \, .$$

is a conditional expectation onto the subalgebra of fixed points of $\mathcal{L}_A^\beta$.

Davies dynamics



Conditional log-Sobolev constant

For $A \subset \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^\beta$ in A by

$$\alpha_\Lambda(\mathcal{L}_A^\beta) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}\left[\mathcal{L}_A^\beta(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)\right]}{2D_A^\beta(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the global evolution (the Gibbs state of a local commuting Hamiltonian), and

$$D_A^\beta(\rho_\Lambda || \sigma_\Lambda) = D(\rho_\Lambda || (\mathcal{E}_A^\beta)^*(\rho_\Lambda)).$$

Davies dynamics

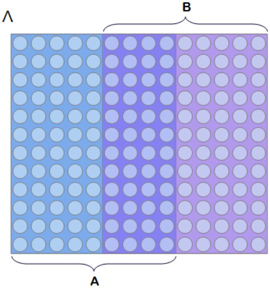
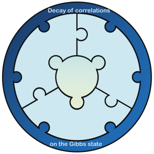


Figure: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

Davies dynamics



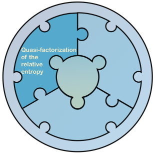
Exponential decay of correlations

If $\sigma \in \mathcal{S}(\mathcal{H})$ is a fixed point of the evolution and $f, g \in \mathcal{A}(\mathcal{H})$ such that $f \in \mathcal{A}_A$ and $g \in \mathcal{A}_B$, then

$$|\mathrm{tr}[\sigma f g] - \mathrm{tr}[\sigma f]\,\mathrm{tr}[\sigma g]| \leq c \|f\|_\infty \|g\|_\infty \, \mathrm{e}^{-d(A\setminus B, B\setminus A)} .$$

Spectral gap	**Log-Sobolev constant**
Change $\|\cdot\|_\infty \mapsto \|\cdot\|_{2,\sigma}$	Change $\|\cdot\|_\infty \mapsto \|\cdot\|_{1,\sigma}$
Conditional version	Conditional version
	Assume it for every fixed point

Davies dynamics

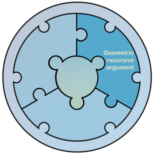


Quasi-factorization (Davies)

Assume that there exists a constant $0 < c < \dfrac{1}{2(4+\sqrt{2})}$ such that there is exponential conditional $\mathbb{L}_1$-clustering of correlations with corresponding constant c. Then, the following inequality holds for every $\rho \in \mathcal{S}(\mathcal{H})$:

$$D^{\beta}_{AB}(\rho||\sigma) \leq \frac{1}{1-2(4+\sqrt{2})\, c} \left(D^{\beta}_{A}(\rho||\sigma) + D^{\beta}_{B}(\rho||\sigma) \right), \qquad (1)$$

for every $\sigma = \mathcal{E}^*_{AB}(\sigma)$.



Geometric recursive argument (Davies)

$$\alpha\left(\mathcal{L}_\Lambda^{\beta *}\right) \geq \Psi(L_0) \min_{R \in \mathcal{R}_{L_0}} \alpha_\Lambda\left(\mathcal{L}_R^{\beta *}\right),$$

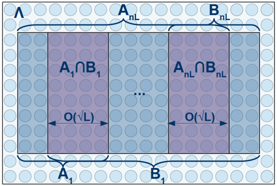
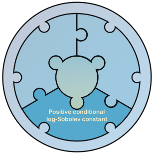


Figure: Splitting in A_n and B_n.



Lemma

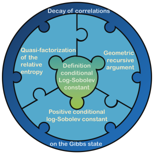
Given $\Lambda \subset\subset \mathbb{Z}^d$, $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ the Lindbladian associated to the Davies dynamics and a finite lattice and $A \subset \Lambda$, we have

$$\alpha_\Lambda \left(\mathcal{L}_A^{\beta *} \right) \geq \psi(|A|) > 0,$$

where $\psi(|A|)$ might depend on Λ, but is independent of its size.

Uses Junge et al. '19.

Davies dynamics



Theorem (Davies)

Under exponential conditional $\mathbb{L}_1$-clustering of correlations, for a k-local commuting Hamiltonian, the Davies dynamics has a positive log-Sobolev constant.

Previous results:

- **Kastoryano-Brandao, '15.** Under strong clustering, for a k-local commuting Hamiltonian, the Davies dynamics is gapped.

2.4 Part 4: A strengthened DPI for the BS-entropy

Main concepts

Relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **relative entropy** is defined as:

$$D(\sigma||\rho) := \mathrm{tr}[\sigma(\log\sigma - \log\rho)].$$

Belavkin-Staszewski relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **BS-entropy** is defined as:

$$D_{\mathrm{BS}}(\sigma||\rho) := \mathrm{tr}\left[\sigma\log\left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right].$$

Main concepts

Relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **relative entropy** is defined as:

$$D(\sigma||\rho) := \operatorname{tr}[\sigma(\log \sigma - \log \rho)].$$

Belavkin-Staszewski relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **BS-entropy** is defined as:

$$D_{\mathrm{BS}}(\sigma||\rho) := \operatorname{tr}\left[\sigma \log \left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right].$$

Relation between relative entropies

The following holds for every $\sigma > 0, \rho > 0$:

$$D_{\mathrm{BS}}(\sigma||\rho) \geq D(\sigma||\rho).$$

Main concepts

Relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **relative entropy** is defined as:

$$D(\sigma||\rho) := \operatorname{tr}[\sigma(\log \sigma - \log \rho)].$$

Belavkin-Staszewski relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **BS-entropy** is defined as:

$$D_{\mathrm{BS}}(\sigma||\rho) := \operatorname{tr}\left[\sigma \log \left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right].$$

Relation between relative entropies

The following holds for every $\sigma > 0, \rho > 0$:

$$D_{\mathrm{BS}}(\sigma||\rho) \geq D(\sigma||\rho).$$

Motivation: Relative entropy

Relative entropy of σ and ρ: $D(\sigma||\rho) := \operatorname{tr}[\sigma(\log \sigma - \log \rho)]$.

Motivation: Relative entropy

Relative entropy of σ and ρ: $D(\sigma||\rho) := \operatorname{tr}[\sigma(\log\sigma - \log\rho)]$.

Data processing inequality

$$D(\sigma||\rho) \geq D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)).$$

Motivation: Relative entropy

Relative entropy of σ and ρ: $D(\sigma||\rho) := \operatorname{tr}[\sigma(\log\sigma - \log\rho)]$.

Data processing inequality

$$D(\sigma||\rho) \geq D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)).$$

Conditions for equality, Petz 1986

$$D(\sigma||\rho) = D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \Leftrightarrow \sigma = \rho^{1/2}\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1/2}\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1/2}\right)\rho^{1/2}.$$

Petz recovery map $\mathcal{R}^{\rho}_{\mathcal{T}}(\cdot) := \rho^{1/2}\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1/2}(\cdot)\mathcal{T}(\rho)^{-1/2}\right)\rho^{1/2}$.

Motivation: Relative entropy

Relative entropy of σ and ρ: $D(\sigma||\rho) := \operatorname{tr}[\sigma(\log \sigma - \log \rho)]$.

Data processing inequality

$$D(\sigma||\rho) \geq D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)).$$

Conditions for equality, Petz 1986

$$D(\sigma||\rho) = D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \Leftrightarrow \sigma = \rho^{1/2}\mathcal{T}^* \left(\mathcal{T}(\rho)^{-1/2}\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1/2} \right) \rho^{1/2}.$$

Petz recovery map $\mathcal{R}_{\mathcal{T}}^{\rho}(\cdot) := \rho^{1/2}\mathcal{T}^* \left(\mathcal{T}(\rho)^{-1/2}(\cdot)\mathcal{T}(\rho)^{-1/2} \right) \rho^{1/2}$.

Motivation: Strengthened bounds for DPI of RE

Problem

Can we find a lower bound for the DPI in terms of $D(\sigma || \mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma))$*?*

Answer: It is not possible **(Brandao et al. '15, Fawzi[2] '17)**.

Motivation: Strengthened bounds for DPI of RE

Problem

Can we find a lower bound for the DPI in terms of $D(\sigma \| \mathcal{R}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma))$?

Answer: It is not possible **(Brandao et al. '15, Fawzi^2 '17)**.

(Carlen-Vershynina '17) $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ conditional expectation, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$:

$$D(\sigma\|\rho) - D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8}\right)^4 \|L_\rho R_{\sigma^{-1}}\|_\infty^{-2} \|\mathcal{R}_{\mathcal{E}}^{\sigma}(\rho_{\mathcal{N}}) - \rho\|_1^4.$$

Motivation: Strengthened bounds for DPI of RE

Problem

Can we find a lower bound for the DPI in terms of $D(\sigma || \mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma))$?

Answer: It is not possible **(Brandao et al. '15, Fawzi^2 '17)**.

(Carlen-Vershynina '17) $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ conditional expectation, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$:

$$D(\sigma\|\rho) - D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8}\right)^4 \|L_\rho R_{\sigma^{-1}}\|_\infty^{-2} \|\mathcal{R}^\sigma_{\mathcal{E}}(\rho_{\mathcal{N}}) - \rho\|_1^4.$$

(Carlen-Vershynina '18) Extension to standard f-divergences.

Motivation: Strengthened bounds for DPI of RE

Problem

Can we find a lower bound for the DPI in terms of $D(\sigma \| \mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma))$*?*

Answer: It is not possible **(Brandao et al. '15, Fawzi[2] '17)**.

(Carlen-Vershynina '17) $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ conditional expectation, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$:

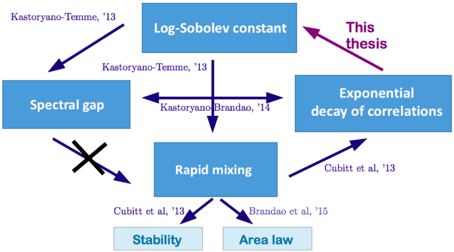
$$D(\sigma\|\rho) - D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8}\right)^4 \|L_\rho R_{\sigma^{-1}}\|_\infty^{-2} \|\mathcal{R}^{\sigma}_{\mathcal{E}}(\rho_{\mathcal{N}}) - \rho\|_1^4 .$$

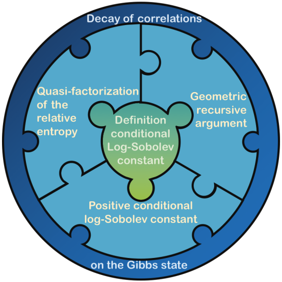
(Carlen-Vershynina '18) Extension to standard f-divergences.

Our results (BS-entropy)

Relative entropy	**BS-entropy**
$\operatorname{tr}[\sigma(\log\sigma - \log\rho)]$	$\operatorname{tr}\left[\sigma\log\left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right]$
$\rho = \rho^{1/2}\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1/2}\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1/2}\right)\rho^{1/2}$	$\sigma = \rho\,\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1}\mathcal{T}(\sigma)\right)$
$\left(\frac{\pi}{8}\right)^4 \Vert L_\rho R_{\sigma^{-1}}\Vert_\infty^{-2}\left\Vert\mathcal{R}_{\mathcal{E}}^{\sigma}(\rho_{\mathcal{N}}) - \rho\right\Vert_1^4$	$\left(\frac{\pi}{8}\right)^4 \Vert\Gamma\Vert_\infty^{-4}\left\Vert\sigma^{-1}\right\Vert_\infty^{-2}\left\Vert\rho - \sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\right\Vert_2^4$
Extension to standard f-divergences	Extension to maximal f-divergences

Conclusion

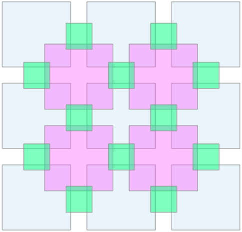

Kastoryano-Temme, '13
Log-Sobolev constant
This thesis
Kastoryano-Temme, '13
Spectral gap
Kastoryano-Brandao, '14
Exponential decay of correlations
Rapid mixing
Cubitt et al, '13
Cubitt et al, '13
Brandao et al, '15
Stability
Area law


Decay of correlations
Quasi-factorization of the relative entropy
Geometric recursive argument
Definition conditional Log-Sobolev constant
Positive conditional log-Sobolev constant
on the Gibbs state

Extension of log-Sobolev for heat-bath to larger dimensions

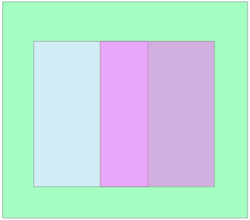
2 possible approaches:

- $D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left(D_A + D_B + D_C\right) (\rho_{ABC}||\sigma_{ABC})$



Extension of log-Sobolev for heat-bath to larger dimensions

- $D_{AB}(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC})\,(D_A(\rho_{ABC}||\sigma_{ABC}) + D_B(\rho_{ABC}||\sigma_{ABC}))$



Possible extensions of this thesis

1. Examples of systems that satisfy clustering of correlations (Davies).
2. Weaken assumptions to obtain log-Sobolev constants.
3. Look for other classes of systems to which we can apply these results.
4. Understand differences between conditions of clustering of correlations (Davies).

Applications

1. Noisy quantum circuits.
2. Mixing rates of divergences.
3. Quantum capacities of channels.

Relative entropy

Properties of the relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** $\rho_{AB} \mapsto D(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity.** $D(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
4. **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB}))$ for every quantum channel $\mathcal{T}$.

Characterization of the RE, Wilming et al. '17, Matsumoto '10

If $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$ satisfies $1 - 4$, then f is the relative entropy.

Relative entropy

Properties of the relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** $\rho_{AB} \mapsto D(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity.** $D(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
4. **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB}))$ for every quantum channel $\mathcal{T}$.

Characterization of the RE, Wilming et al. '17, Matsumoto '10

If $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$ satisfies $1-4$, then f is the relative entropy.

Conditional relative entropy

Conditional relative entropy, (Q-Fact)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a **conditional relative entropy** in A as a function

$$D_A(\cdot||\cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and

   (2.1) $D_A(\rho_{AB}||\sigma_{AB})$=0 if, and only if, $\rho_{AB} = \sigma_{AB}^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma_{AB}^{1/2}$.
3. **Semi-superadditivity:** $D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$ and

   (3.1) **Semi-additivity:** if $\rho_{AB} = \rho_A \otimes \rho_B$,
   $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.
4. **Semi-motonicity:** For every quantum channel $\mathcal{T}$,

$$D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) + D_B((\operatorname{tr}_A \circ \mathcal{T})(\rho_{AB})||(\operatorname{tr}_A \circ \mathcal{T})(\sigma_{AB}))$$
$$\leq D_A(\rho_{AB}||\sigma_{AB}) + D_B(\operatorname{tr}_A(\rho_{AB})|| \operatorname{tr}_A(\sigma_{AB})).$$

Remark

Consider for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$

$$D^{+}_{A,B}(\rho_{AB}||\sigma_{AB}) = D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}).$$

Then, $D^{+}_{A,B}$ verifies the following properties:

1. **Continuity:** $\rho_{AB} \mapsto D^{+}_{A,B}(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity:** $D^{+}_{A,B}(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity:** $D^{+}_{A,B}(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

However, it does not satisfy the property of monotonicity.

Weak conditional relative entropy

Weak conditional relative entropy, (Q-Fact)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a **conditional relative entropy** in A as a function

$$D_A(\cdot||\cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and
   (2.1) $D_A(\rho_{AB}||\sigma_{AB})=0$ if, and only if, $\rho_{AB} = \sigma_{AB}^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma_{AB}^{1/2}$.
3. **Semi-superadditivity:** $D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$ and
   (3.1) **Semi-additivity:** if $\rho_{AB} = \rho_A \otimes \rho_B$,
   $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.

Conditional relative entropy by expectations

Conditional relative entropy by expectations, (Q-Fact)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in S_{AB}$. Let $\mathbb{E}_A^*$ be defined as

$$\mathbb{E}_A^*(\rho_{AB}) := \sigma_{AB}^{1/2}\, \sigma_B^{-1/2}\, \rho_B\, \sigma_B^{-1/2}\, \sigma_{AB}^{1/2}. \tag{2}$$

We define the **conditional relative entropy by expectations** of ρ_{AB} and σ_{AB} in A by:

$$D_A^E(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\mathbb{E}_A^*(\rho_{AB})).$$

Property

$D_A^E(\rho_{AB}||\sigma_{AB})$ is a weak conditional relative entropy.

Quasi-factorization CRE by expectations, **(Q-Fact)**

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following inequality holds

$$(1 - \xi(\sigma_{AB}))D(\rho_{AB}||\sigma_{AB}) \leq D_A^E(\rho_{AB}||\sigma_{AB}) + D_B^E(\rho_{AB}||\sigma_{AB}), \tag{3}$$

where

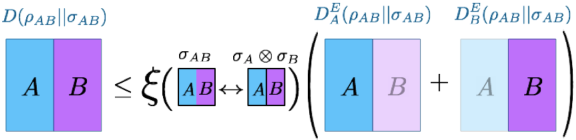
$$\xi(\sigma_{ABC}) = 2\,(E_1(t) + E_2(t)),$$

and

$$E_1(t) = \int_{-\infty}^{+\infty} dt\, \beta_0(t) \left\| \sigma_B^{\frac{-1+it}{2}} \sigma_{AB}^{\frac{1-it}{2}} \sigma_A^{\frac{-1+it}{2}} - \mathbb{1}_{AB} \right\|_\infty \left\| \sigma_A^{-1/2} \sigma_{AB}^{\frac{1+it}{2}} \sigma_B^{-1/2} \right\|_\infty,$$

$$E_2(t) = \int_{-\infty}^{+\infty} dt\, \beta_0(t) \left\| \sigma_B^{\frac{-1-it}{2}} \sigma_{AB}^{\frac{1+it}{2}} \sigma_A^{\frac{-1-it}{2}} - \mathbb{1}_{AB} \right\|_\infty .$$

Note that $\xi(\sigma_{AB}) = 0$ if σ_{AB} is a tensor product between A and B.



Relation with the classical case

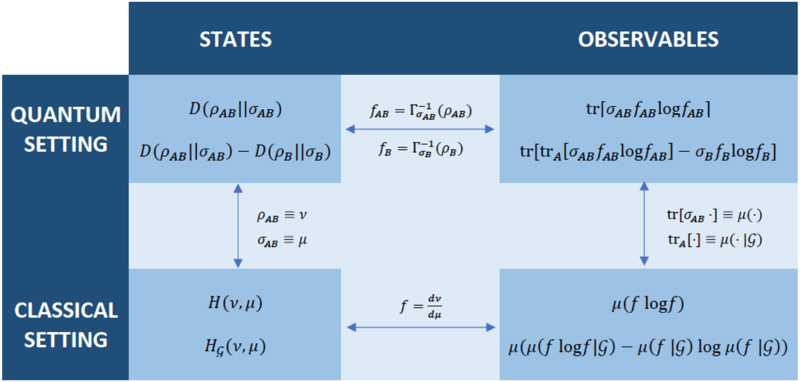


Figure: Identification between classical and quantum quantities when the states considered are classical.

Standard and maximal f-divergences

(Hiai-Mosonyi '17)

Standard f-divergences

Let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$S_f(\sigma \| \rho) = \operatorname{tr}\left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) \rho^{1/2}\right]$$

is the *standard f-divergence.*

Standard and maximal f-divergences

(Hiai-Mosonyi '17)

Standard f-divergences

Let $f : (0,\infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$S_f(\sigma\|\rho) = \operatorname{tr}\left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}})\rho^{1/2}\right]$$

is the *standard f-divergence.*

Maximal f-divergences

Let $f : (0,\infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$\hat{S}_f(\sigma\|\rho) = \operatorname{tr}\left[\rho^{1/2} f(\rho^{-1/2}\sigma\rho^{-1/2})\rho^{1/2}\right]$$

is the *maximal f-divergence.*

Standard and maximal f-divergences

(Hiai-Mosonyi '17)

Standard f-divergences

Let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$S_f(\sigma \| \rho) = \operatorname{tr}\left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) \rho^{1/2}\right]$$

is the *standard f-divergence.*

Maximal f-divergences

Let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$\hat{S}_f(\sigma \| \rho) = \operatorname{tr}\left[\rho^{1/2} f(\rho^{-1/2} \sigma \rho^{-1/2}) \rho^{1/2}\right]$$

is the *maximal f-divergence.*

Data processing inequality

Let $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a PTP linear map. Then,

$$\hat{S}_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) \leq \hat{S}_f(\sigma \| \rho).$$

Relation between f-divergences

For every two states $\sigma > 0$, $\rho > 0$ on $\mathcal{M}$ and every operator convex function $f : (0, \infty) \to \mathbb{R}$,

$$S_f(\sigma \| \rho) \leq \hat{S}_f(\sigma \| \rho).$$

Data processing inequality

Let $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a PTP linear map. Then,

$$\hat{S}_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) \leq \hat{S}_f(\sigma \| \rho).$$

Relation between f-divergences

For every two states $\sigma > 0$, $\rho > 0$ on $\mathcal{M}$ and every operator convex function $f : (0, \infty) \to \mathbb{R}$,

$$S_f(\sigma \| \rho) \leq \hat{S}_f(\sigma \| \rho).$$

Remark: Difference

For maximal f-divergences, there is no equivalent condition for equality in DPI which provides a explicit expression of recovery for σ.

Data processing inequality

Let $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a PTP linear map. Then,

$$\hat{S}_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) \leq \hat{S}_f(\sigma \| \rho).$$

Relation between f-divergences

For every two states $\sigma > 0$, $\rho > 0$ on $\mathcal{M}$ and every operator convex function $f : (0, \infty) \to \mathbb{R}$,

$$S_f(\sigma \| \rho) \leq \hat{S}_f(\sigma \| \rho).$$

Remark: Difference

For maximal f-divergences, there is no equivalent condition for equality in DPI which provides a explicit expression of recovery for σ.

Equivalent conditions for equality on DPI

$\Gamma := \sigma^{-1/2} \rho \sigma^{-1/2}$ and $\Gamma_{\mathcal{N}} := \sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2}$
$\rho_{\mathcal{N}} := \mathcal{E}(\rho)$, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$

Equivalent conditions for equality on DPI (BS-entropy)

Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on $\mathcal{M}$. Then, the following are equivalent:

1. $\hat{S}_{\text{BS}}(\sigma \| \rho) = \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}})$.
2. $\rho_{\mathcal{N}} \rho^{-1} = \sigma_{\mathcal{N}} \sigma^{-1}$.
3. $\sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} = \Gamma^{1/2} \sigma^{1/2}$.

Equivalent conditions for equality on DPI

$\Gamma := \sigma^{-1/2}\rho\sigma^{-1/2}$ and $\Gamma_{\mathcal{N}} := \sigma_{\mathcal{N}}^{-1/2}\rho_{\mathcal{N}}\sigma_{\mathcal{N}}^{-1/2}$
$\rho_{\mathcal{N}} := \mathcal{E}(\rho)$, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$

Equivalent conditions for equality on DPI **(BS-entropy)**

Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on $\mathcal{M}$. Then, the following are equivalent:

1. $\hat{S}_{\text{BS}}(\sigma\|\rho) = \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}})$.
2. $\rho_{\mathcal{N}}\rho^{-1} = \sigma_{\mathcal{N}}\sigma^{-1}$.
3. $\sigma^{1/2}\sigma_{\mathcal{N}}^{-1/2}\Gamma_{\mathcal{N}}^{1/2}\sigma_{\mathcal{N}}^{1/2} = \Gamma^{1/2}\sigma^{1/2}$.

BS recovery condition, **(BS-entropy)**

$$\mathcal{T}_{\mathcal{E}}^{\sigma}(\cdot) := \sigma\sigma_{\mathcal{N}}^{-1}(\cdot).$$

Equivalent conditions for equality on DPI

$\Gamma := \sigma^{-1/2} \rho \sigma^{-1/2}$ and $\Gamma_{\mathcal{N}} := \sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2}$
$\rho_{\mathcal{N}} := \mathcal{E}(\rho)$, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$

Equivalent conditions for equality on DPI (BS-entropy)

Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on $\mathcal{M}$. Then, the following are equivalent:

1. $\hat{S}_{\mathrm{BS}}(\sigma \| \rho) = \hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}})$.
2. $\rho_{\mathcal{N}} \rho^{-1} = \sigma_{\mathcal{N}} \sigma^{-1}$.
3. $\sigma^{1/2} \sigma_{\mathcal{N}}^{-1/2} \Gamma_{\mathcal{N}}^{1/2} \sigma_{\mathcal{N}}^{1/2} = \Gamma^{1/2} \sigma^{1/2}$.

BS recovery condition, (BS-entropy)

$$\mathcal{T}_{\mathcal{E}}^{\sigma}(\cdot) := \sigma \sigma_{\mathcal{N}}^{-1}(\cdot).$$

Consequences

Note: Although they can be seen as a consequence of the previous result, the following facts were previously known.

Corollary

$$\begin{aligned}\hat{S}_{\mathrm{BS}}(\sigma\|\rho) = \hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) &\Leftrightarrow \rho = \mathcal{T}_{\mathcal{E}}^{\sigma} \circ \mathcal{E}(\rho) \\ &\Leftrightarrow \sigma = \mathcal{T}_{\mathcal{E}}^{\rho} \circ \mathcal{E}(\sigma) \\ &\Leftrightarrow \hat{S}_{\mathrm{BS}}(\rho\|\sigma) = \hat{S}_{\mathrm{BS}}(\rho_{\mathcal{N}}\|\sigma_{\mathcal{N}}).\end{aligned}$$

Consequences

Note: Although they can be seen as a consequence of the previous result, the following facts were previously known.

Corollary

$$\begin{aligned}\hat{S}_{\mathrm{BS}}(\sigma\|\rho) = \hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) &\Leftrightarrow \rho = \mathcal{T}_{\mathcal{E}}^{\sigma} \circ \mathcal{E}(\rho) \\ &\Leftrightarrow \sigma = \mathcal{T}_{\mathcal{E}}^{\rho} \circ \mathcal{E}(\sigma) \\ &\Leftrightarrow \hat{S}_{\mathrm{BS}}(\rho\|\sigma) = \hat{S}_{\mathrm{BS}}(\rho_{\mathcal{N}}\|\sigma_{\mathcal{N}}).\end{aligned}$$

Corollary

$$D(\sigma\|\rho) = D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \implies \hat{S}_{BS}(\sigma\|\rho) = \hat{S}_{BS}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}).$$

Equivalently,

$$\sigma = \mathcal{R}_{\mathcal{E}}^{\rho} \circ \mathcal{E}(\sigma) \implies \sigma = \mathcal{T}_{\mathcal{E}}^{\rho} \circ \mathcal{E}(\sigma).$$

The converse of this result is false (**Jencová-Petz-Pitrik '09, Hiai-Mosonyi '17**).

Consequences

Note: Although they can be seen as a consequence of the previous result, the following facts were previously known.

Corollary

$$\begin{aligned}\hat{S}_{\mathrm{BS}}(\sigma\|\rho) = \hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) &\Leftrightarrow \rho = \mathcal{T}_{\mathcal{E}}^{\sigma} \circ \mathcal{E}(\rho) \\ &\Leftrightarrow \sigma = \mathcal{T}_{\mathcal{E}}^{\rho} \circ \mathcal{E}(\sigma) \\ &\Leftrightarrow \hat{S}_{\mathrm{BS}}(\rho\|\sigma) = \hat{S}_{\mathrm{BS}}(\rho_{\mathcal{N}}\|\sigma_{\mathcal{N}}).\end{aligned}$$

Corollary

$$D(\sigma\|\rho) = D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \implies \hat{S}_{BS}(\sigma\|\rho) = \hat{S}_{BS}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}).$$

Equivalently,

$$\sigma = \mathcal{R}_{\mathcal{E}}^{\rho} \circ \mathcal{E}(\sigma) \implies \sigma = \mathcal{T}_{\mathcal{E}}^{\rho} \circ \mathcal{E}(\sigma).$$

The converse of this result is false (**Jencová-Petz-Pitrik '09, Hiai-Mosonyi '17**).

Strengthened DPI for the BS-entropy

Strengthened DPI for the BS-entropy (BS-entropy)

Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states onto $\mathcal{M}$. Then,

$$\hat{S}_{\text{BS}}(\sigma\|\rho) - \hat{S}_{\text{BS}}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8}\right)^4 \|\Gamma\|_\infty^{-4} \left\|\sigma^{-1}\right\|_\infty^{-2} \left\|\rho - \sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\right\|_2^4 .$$

Strenghtened DPI for maximal f-divergences

Strenghtened DPI for maximal f-divergences (BS-entropy)

Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on $\mathcal{M}$ and let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function with transpose $\tilde{f}$. We assume that $\tilde{f}$ is operator monotone decreasing and such that the measure $\mu_{-\tilde{f}}$ that appears in the representation of $-\tilde{f}$ is absolutely continuous with respect to Lebesgue measure. Moreover, we assume that for every $T \geq 1$, there exist constants $\alpha \geq 0$, $C > 0$ satisfying $\mathrm{d}\mu_{-\tilde{f}}(t)/\mathrm{d}t \geq (CT^{2\alpha})^{-1}$ for all $t \in [1/T, T]$ and such that

$$\left(\frac{(2\alpha+1)\sqrt{C}}{4} \frac{(\hat{S}_f(\sigma\|\rho) - \hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}))^{1/2}}{1 + \|\Gamma\|_\infty} \right)^{\frac{1}{1+\alpha}} \leq 1.$$

Then, there is a constant $L_\alpha > 0$ such that

$$\hat{S}_f(\sigma\|\rho) - \hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq$$
$$\geq \frac{L_\alpha}{C} \left(1 + \|\Gamma\|_\infty\right)^{-(4\alpha+2)} \|\Gamma\|_\infty^{-(2\alpha+2)} \left\|\sigma^{-1}\right\|_\infty^{-(2\alpha+2)} \left\|\rho - \sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\right\|_2^{4(\alpha+1)}.$$