Logarithmic Sobolev Inequalities for Quantum Many-Body Systems

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Joint work with Ivan Bardet (INRIA, Paris), Angelo Lucia (Caltech), Cambyse Rouzé (T. U. München) and David Pérez-García (U. Complutense de Madrid).

Perimeter Institute Quantum Discussions, 16th October 2019

Based on:

1. A. Capel, A. Lucia and D. Pérez-García, **Superadditivity of Quantum Relative Entropy for General States**, *IEEE Trans. on Inf. Theory*, 64 (7) (2018), 4758–4765.

2. A. Capel, A. Lucia and D. Pérez-García, **Quantum Conditional Relative Entropy and Quasi-Factorization of the Relative Entropy**, *J. Phys. A: Math. Theor.*, 51 (2018), 484001.

3. I. Bardet, A. Capel, A. Lucia, D. Pérez-García and C. Rouzé, **On the modified logarithmic Sobolev inequality for the heat-bath dynamics for 1D systems**, preprint, arXiv: 1908.09004.

4. I. Bardet, A. Capel and C. Rouzé, **Positivity of the modified logarithmic Sobolev constant for quantum Davies semigroups: the commuting case**, in preparation.

Quantum

Q. information theory $\longleftrightarrow$ Q. many-body physics

Communication channels $\longleftrightarrow$ Physical interactions

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Main topic of this talk

Field of study

Dissipative evolutions of quantum many-body systems

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Velocity of convergence of certain quantum dissipative evolutions to their thermal equilibriums.

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Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant.

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Contents

1. Quantum dissipative systems

Open quantum systems

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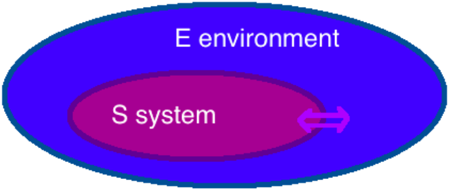
$\Rightarrow$ Open quantum many-body systems.



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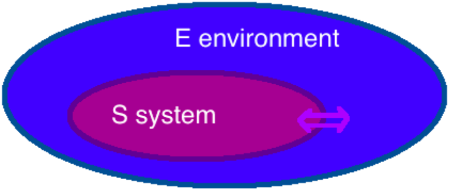

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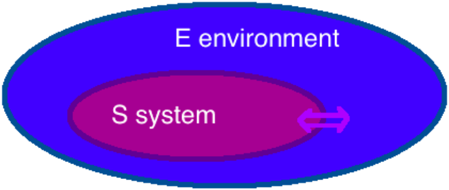

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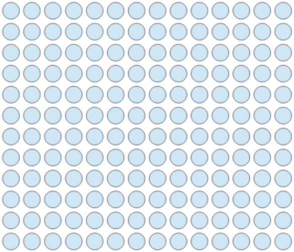

Figure: A quantum spin lattice system.

- Finite lattice $\Lambda \subset\subset \mathbb{Z}^d$.
- To every site $x \in \Lambda$ we associate $\mathcal{H}_x$ $(= \mathbb{C}^D)$.
- The global Hilbert space associated to Λ is $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$.
- The set of bounded linear endomorphisms on $\mathcal{H}_\Lambda$ is denoted by $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$.
- The set of density matrices is denoted by $\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{\rho_\Lambda \in \mathcal{B}_\Lambda \ : \ \rho_\Lambda \geq 0 \text{ and } \operatorname{tr}[\rho_\Lambda] = 1\}$.

Evolution of a system

Isolated system.

Physical evolution: $\rho \mapsto U\rho U^*$ $\rightsquigarrow$ Reversible

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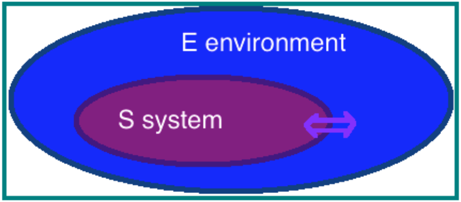

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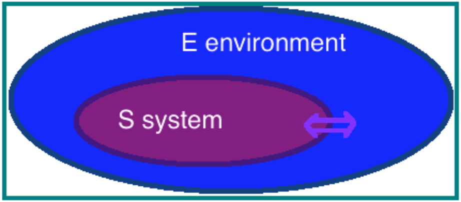

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$$\rho \mapsto \rho \otimes |\psi\rangle \langle\psi|_E \mapsto U \left(\rho \otimes |\psi\rangle \langle\psi|_E\right) U^* \mapsto \operatorname{tr}_E[U \left(\rho \otimes |\psi\rangle \langle\psi|_E\right) U^*] = \tilde{\rho}$$

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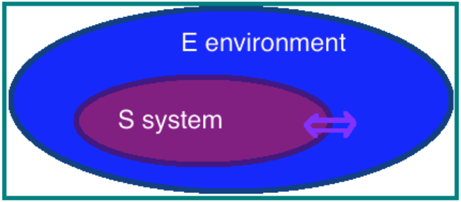

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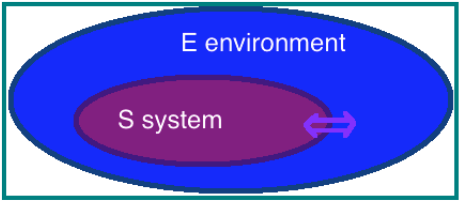

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Continuous-time description: For every $t \geq 0$, the corresponding time slice is a realizable evolution $\mathcal{T}_t$ (quantum channel).

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A **dissipative quantum system** is a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

Semigroup:

- $\mathcal{T}_t^* \circ \mathcal{T}_s^* = \mathcal{T}_{t+s}^*$.
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QMS generator

The infinitesimal generator $\mathcal{L}_\Lambda^*$ of the previous semigroup of quantum channels is usually called **Liouvillian**, or **Lindbladian**.

$$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*} \Leftrightarrow \mathcal{L}_\Lambda^* = \frac{d}{dt}\mathcal{T}_t^* \mid_{t=0} .$$

Notation: $\rho_t := \mathcal{T}_t^*(\rho)$.

$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t^*(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$

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Recent change of perspective ⇒ Resource to exploit

New area:

Quantum dissipative engineering,

to create artificial evolutions in which the dissipative process works in favor (protecting the system from noisy evolutions).

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We define the **mixing time** of $\{\mathcal{T}_t^*\}$ by

$$\tau(\varepsilon) = \min\left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \mathcal{T}_t^*(\rho) - \mathcal{T}_\infty^*(\rho) \right\|_1 \leq \varepsilon \right\}.$$

Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

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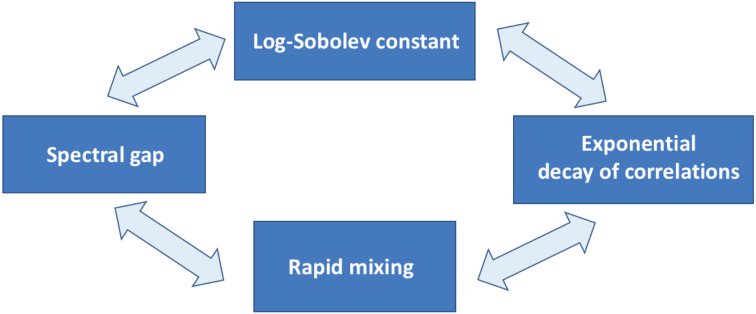

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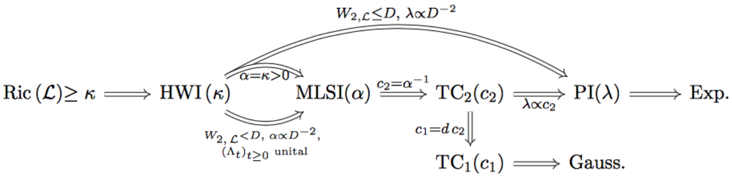

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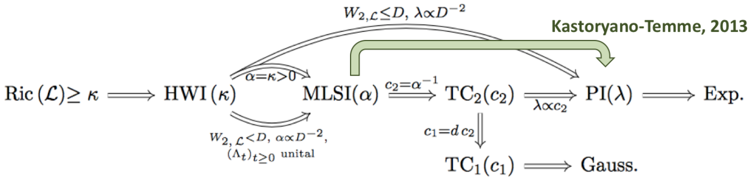

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Main problem of this talk

Develop a strategy to find positive log Sobolev constants.

Concrete problem

Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant.

Main problem of this talk

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Classical spin systems

(Cesi, Dai Pra-Paganoni-Posta, '02)

(1) Quasi-factorization of the entropy (in terms of a conditional entropy).

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$\Downarrow$

Positive log-Sobolev constant.

Conditional log-Sobolev constant

Log-Sobolev constant

Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ. We define the **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ by

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

Conditional log-Sobolev constant

Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ, $A \subseteq \Lambda$. We define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ on A by

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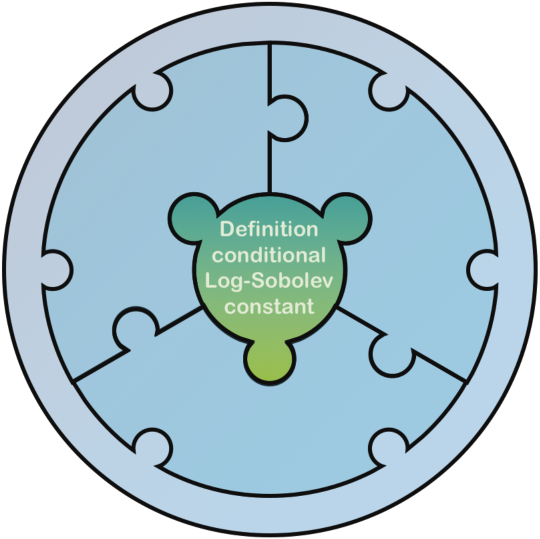
Strategy



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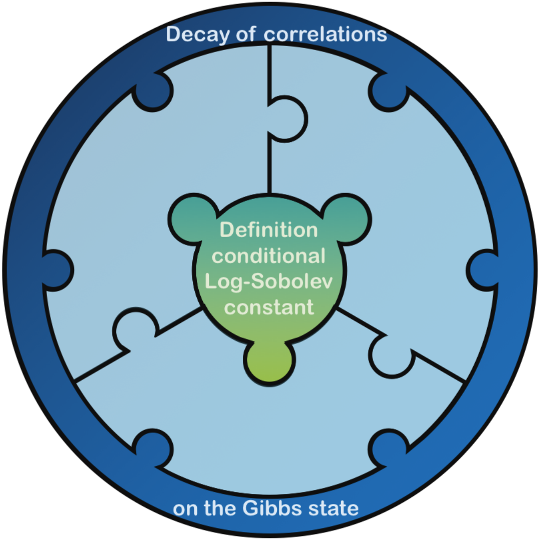
Decay of correlations
Definition conditional Log-Sobolev constant
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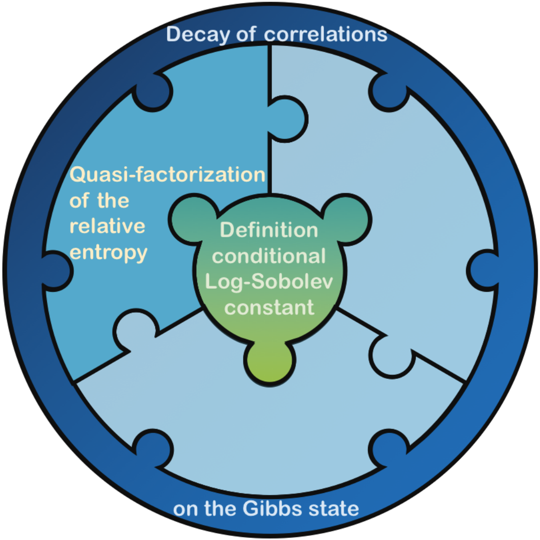
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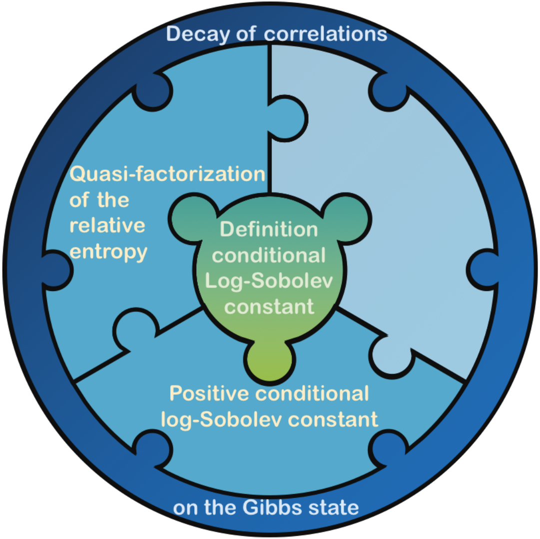
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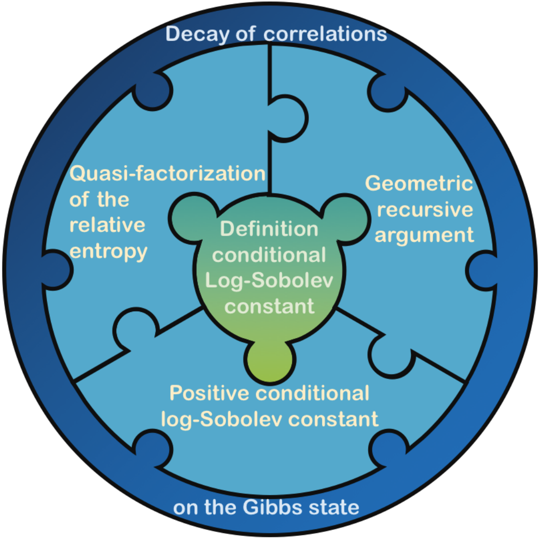
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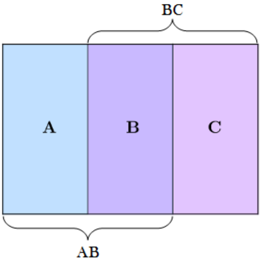
2. Quasi-factorization of the relative entropy

Statement of the problem



Problem

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in S_{ABC}$. Can we prove something like

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right] \; ?$$

Quantum relative entropy

$$D(\rho||\sigma) = \operatorname{tr}\left[\rho(\log \rho - \log \sigma)\right]$$

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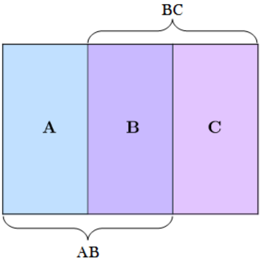

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Classical case, Dai Pra et al. '02

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{1 - 4\|h-1\|_\infty} \, \mu \left[\mathrm{Ent}_\mu(f \mid \mathcal{F}_1) + \mathrm{Ent}_\mu(f \mid \mathcal{F}_2) \right],$$

where $h = \dfrac{d\mu}{d\bar{\mu}}$.

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Classical entropy and conditional entropy

Entropy:

$$\text{Ent}_\mu(f) = \mu(f \log f) - \mu(f)\log \mu(f).$$

Conditional entropy:

$$\text{Ent}_\mu(f \mid \mathcal{G}) = \mu(f \log f \mid \mathcal{G}) - \mu(f \mid \mathcal{G})\log \mu(f \mid \mathcal{G}).$$

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Relative entropy

Quantum relative entropy

Let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$. The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

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Properties of the relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

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Characterization of the RE, Wilming et al. '17, Matsumoto '10

If $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$ satisfies $1-4$, then f is the relative entropy.

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Conditional relative entropy

Conditional relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a **conditional relative entropy** in A as a function

$$D_A(\cdot||\cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and
   (2.1) $D_A(\rho_{AB}||\sigma_{AB})=0$ if, and only if, $\rho_{AB} = \sigma_{AB}^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma_{AB}^{1/2}$.
3. **Semi-superadditivity:** $D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$ and
   (3.1) **Semi-additivity:** if $\rho_{AB} = \rho_A \otimes \rho_B$,
   $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.
4. **Semi-motonicity:** For every quantum channel $\mathcal{T}$,

$$D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) + D_B((\operatorname{tr}_A \circ \mathcal{T})(\rho_{AB})||(\operatorname{tr}_A \circ \mathcal{T})(\sigma_{AB}))$$
$$\leq D_A(\rho_{AB}||\sigma_{AB}) + D_B(\operatorname{tr}_A(\rho_{AB})||\operatorname{tr}_A(\sigma_{AB})).$$

Remark

Consider for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$

$$D^+_{A,B}(\rho_{AB}||\sigma_{AB}) = D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}).$$

Then, $D^+_{A,B}$ verifies the following properties:

1. **Continuity:** $\rho_{AB} \mapsto D^+_{A,B}(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity:** $D^+_{A,B}(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity:** $D^+_{A,B}(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

However, it does not satisfy the property of monotonicity.

Axiomatic characterization of the CRE (C-Lucia-Pérez García, '18)

The only possible conditional relative entropy is given by:

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B)$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

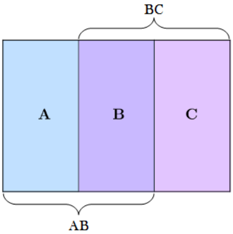

Figure: Choice of indices in $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Result of **quasi-factorization** of the relative entropy, for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$:

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

Quasi-factorization for the CRE (C-Lucia-Pérez García, '18)

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \frac{1}{1 - 2\|H(\sigma_{AC})\|_\infty} \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \sigma_{AC} \, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Note that $H(\sigma_{AC}) = 0$ if σ_{AC} is a tensor product between A and C.

$$(1-2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \leq$$
$$D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) =$$
$$= 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).$$

$$\Leftrightarrow$$

$$(1+2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

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This result is equivalent to:

$$\boxed{(1 + 2\|H(\sigma_{AB})\|_\infty) D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

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Due to:

- **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

we have

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

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Relation with the classical case

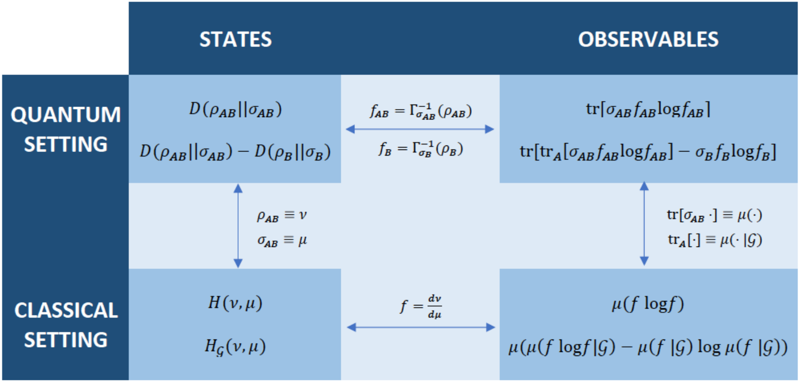

Figure: Identification between classical and quantum quantities when the states considered are classical.

3. Log-Sobolev constant

Quantum spin lattices

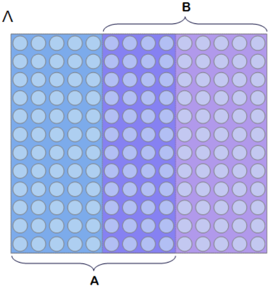

Figure: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

Problem

For a certain $\mathcal{L}_\Lambda^*$, can we prove $\alpha(\mathcal{L}_\Lambda^*) > 0$ using the result of quasi-factorization of the relative entropy?

Example 1

Heat-bath dynamics with tensor product fixed point

Heat-bath with tensor product fixed point

Theorem (C-Lucia-Pérez García, '18)

The **heat-bath dynamics**, with tensor product fixed point, has a positive log-Sobolev constant.

Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x \in \Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

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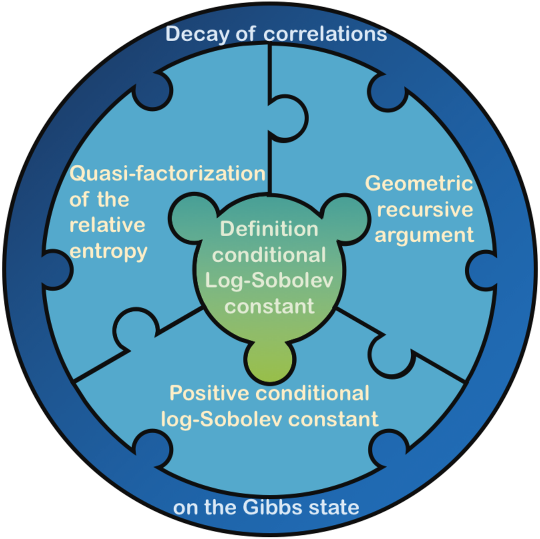
Strategy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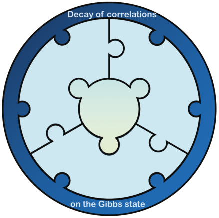
Heat-bath with tensor product fixed point

Assumption

$$\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x .$$



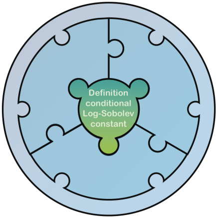
Heat-bath with tensor product fixed point

Conditional log-Sobolev constant

For $x \in \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in x by

$$\alpha_\Lambda(\mathcal{L}_x^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_x(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and $D_x(\rho_\Lambda||\sigma_\Lambda)$ is the conditional relative entropy.

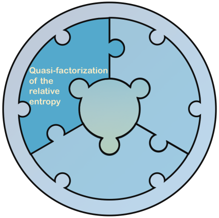


Heat-bath with tensor product fixed point

General quasi-factorization for σ a tensor product

Let $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ and $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x\in\Lambda} \sigma_x$. The following inequality holds:

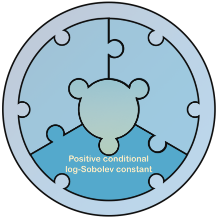
$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x\in\Lambda} D_x(\rho_\Lambda || \sigma_\Lambda).$$



Heat-bath with tensor product fixed point

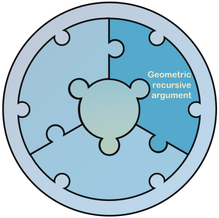
Lemma (Positivity of the conditional log-Sobolev constant)

$$\alpha_\Lambda(\mathcal{L}_x^*) \geq \frac{1}{2}.$$



Heat-bath with tensor product fixed point

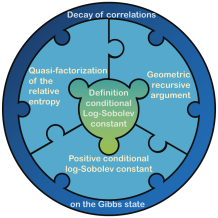
$$
\begin{aligned}
D(\rho_\Lambda || \sigma_\Lambda) &\leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \\
&\leq \sum_{x \in \Lambda} \frac{- \operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2 \alpha_\Lambda(\mathcal{L}_x^*)} \\
&\leq \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x \in \Lambda} - \operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\
&= \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \right) \\
&\leq \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \right) .
\end{aligned}
$$



Heat-bath with tensor product fixed point

Positive log-Sobolev constant

$$\alpha(\mathcal{L}_\Lambda^*) \geq \frac{1}{2}.$$



Example 2

Heat-bath dynamics in 1D

Heat-bath dynamics in 1D

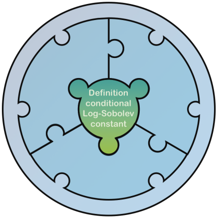
Conditional log-Sobolev constant

For $A \subset \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in A by

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2 D_A(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and

$$D_A(\rho_\Lambda || \sigma_\Lambda) = D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{A^c} || \sigma_{A^c}).$$



Heat-bath dynamics in 1D

Assumption 1

In a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$, A and B not connected, we have

$$\|h(\sigma_{AB})\|_\infty = \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty \leq K < \frac{1}{2}.$$

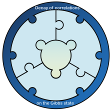
In particular, classical Gibbs states satisfy this.

Assumption 2

For any $B \subset \Lambda$, $B = B_1 \cup B_2$, it holds:

$$D_B(\rho_\Lambda || \sigma_\Lambda) \leq f(\sigma_{B\partial}) \left(D_{B_1}(\rho_\Lambda || \sigma_\Lambda) + D_{B_2}(\rho_\Lambda || \sigma_\Lambda) \right).$$

In particular, tensor products satisfy this (with $f = 1$).



Heat-bath dynamics in 1D

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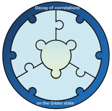
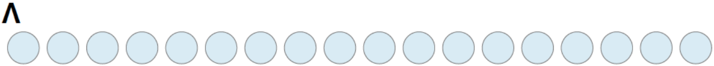


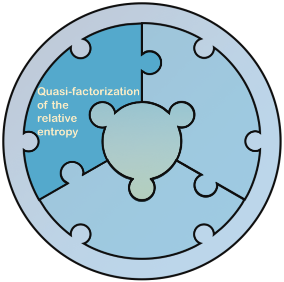
Examples of positive log-Sobolev constants

Theorem (Bardet-C-Lucia-Pérez García-Rouzé, '19)

In 1D, if Assumptions 1 and 2 hold, for a k-local commuting Hamiltonian, the heat-bath dynamics has a positive log-Sobolev constant.

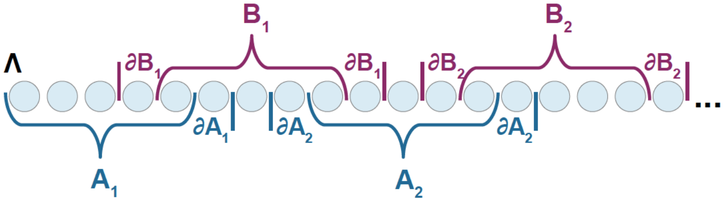
Sketch of the proof





Sketch of the proof

STEP 1



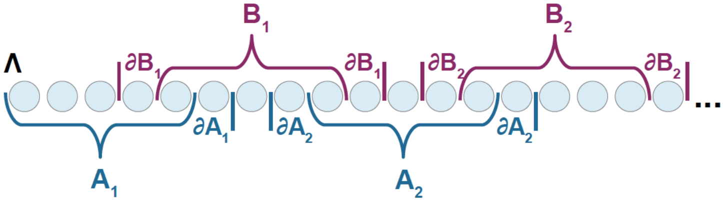
$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{j=1}^{n} B_j$$

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2\| h(\sigma_{A^c B^c}) \|_\infty} \left[D_A(\rho_\Lambda || \sigma_\Lambda) + D_B(\rho_\Lambda || \sigma_\Lambda) \right],$$

$$h(\sigma_{A^c B^c}) := \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^c B^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - \mathbb{1}_{A^c B^c}.$$

Sketch of the proof

STEP 1



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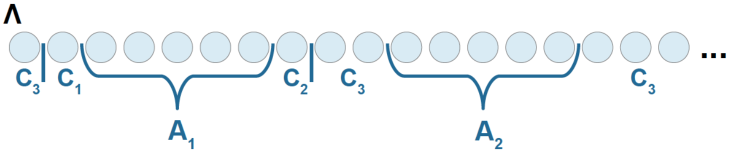
Sketch of the proof

STEP 2



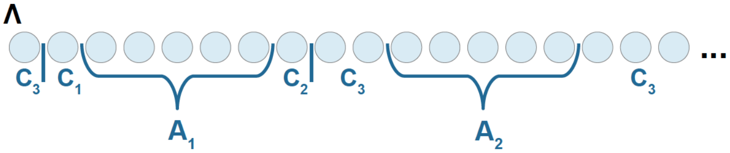
$$D_A(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{i=1}^{n} D_{A_i}(\rho_\Lambda || \sigma_\Lambda)$$

σ_Λ is a QMC between $A_1 \leftrightarrow \partial A_1 \leftrightarrow \Lambda \setminus (A_1 \cup \partial A_1)$

$$\sigma_\Lambda = \bigoplus_{i \in I} \sigma_{A_1 (\partial a_1)_i^L} \otimes \sigma_{(\partial a_1)_i^R \Lambda \setminus (A_1 \cup \partial A_1)}$$

Sketch of the proof

STEP 2



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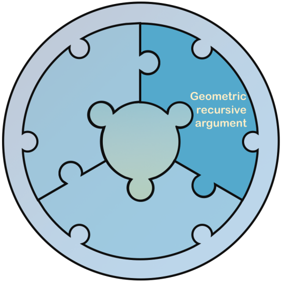
Sketch of the proof

STEP 3

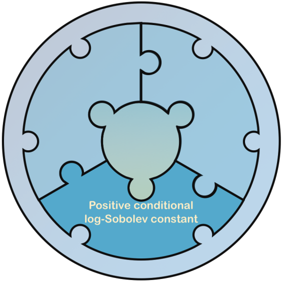
$$\text{Assumption 1} \Rightarrow \alpha(\mathcal{L}_\Lambda^*) \geq \tilde{K} \min_{i \in \{1, \ldots n\}} \{\alpha_\Lambda(\mathcal{L}_{A_i}^*), \alpha_\Lambda(\mathcal{L}_{B_i}^*)\}$$



Sketch of the proof

STEP 4

$$\text{Assumption 2} \Rightarrow \alpha_\Lambda(\mathcal{L}^*_{A_i}) \geq g(\sigma_{A_i\partial}) > 0.$$



Example 3

Davies dynamics

Davies dynamics

Generator

The generator of the Davies dynamics is of the following form:

$$\mathcal{L}_{\Lambda}^{\beta}(X) = i[H_{\Lambda}, X] + \sum_{k \in \Lambda} \mathcal{L}_{k}^{\beta}(X) \,,$$

where

$$\mathcal{L}_{k}^{\beta}(X) = \sum_{\omega, \alpha} \chi_{\alpha,k}^{\beta}(\omega) \left(S_{\alpha,k}^{*}(\omega) X S_{\alpha,k}(\omega) - \frac{1}{2} \left\{ S_{\alpha,k}^{*}(\omega) S_{\alpha,k}(\omega), X \right\} \right) .$$

Important property: Given $A \subseteq \Lambda$,

$$\mathcal{E}_{A}^{\beta}(X) := \mathcal{E}(X|\mathcal{N}) = \lim_{t \to \infty} \mathrm{e}^{t\mathcal{L}_{A}^{\beta}}(X) \,.$$

is a conditional expectation onto the subalgebra of fixed points of $\mathcal{L}_{A}^{\beta}$.

Davies dynamics

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Davies dynamics

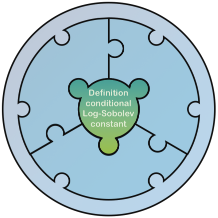
Conditional log-Sobolev constant

For $A \subset \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^\beta$ in A by

$$\alpha_\Lambda(\mathcal{L}_A^\beta) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}\left[\mathcal{L}_A^\beta(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)\right]}{2D_A^\beta(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the global evolution (the Gibbs state of a local commuting Hamiltonian), and

$$D_A^\beta(\rho_\Lambda || \sigma_\Lambda) = D(\rho_\Lambda || \mathcal{E}_A^\beta(\rho_\Lambda)).$$



Davies dynamics

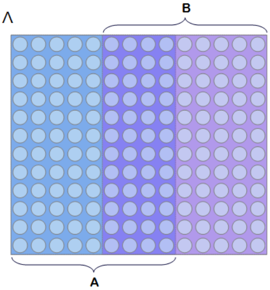


Figure: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

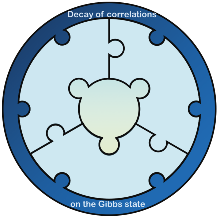
Davies dynamics

Clustering of correlations

The state $\sigma \in \mathcal{S}(\mathcal{H})$ is said to satisfy **exponential conditional $\mathbb{L}_1$-clustering of correlations** with respect to the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB})$ if there exists a constant $c := c(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB}, \sigma)$ such that, for any $X \in \mathcal{B}(\mathcal{H})$,

$$|\operatorname{Cov}_{\mathcal{N}_{AB},\sigma}(\mathcal{E}_A(X), \mathcal{E}_B(X))| \leq c \, \|X\|^2_{\mathbb{L}_1(\sigma)} e^{-d(A \setminus B, B \setminus A)/\xi} .$$

Moreover, the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB})$ is said to satisfy **exponential conditional $\mathbb{L}_1$-clustering of correlations** if there exists a constant $c := c(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB}, \sigma)$ such that any state $\sigma = \mathcal{E}^*_{AB}(\sigma)$ satisfies conditional $\mathbb{L}_1$-clustering of correlations with constant c.



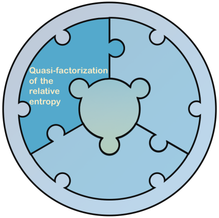
Davies dynamics

Quasi-factorization, Bardet-C-Rouzé '19

Assume that there exists a constant $0 < c < \frac{1}{2(4+\sqrt{2})}$ such that the triple $(\mathcal{N}_A, \mathcal{N}_B, \mathcal{N}_{AB})$ satisfies the exponential conditional $\mathbb{L}_1$-clustering of correlations with corresponding constant c. Then, the following inequality holds for every $\rho \in \mathcal{S}(\mathcal{H})$:

$$D^{\beta}_{AB}(\rho||\sigma) \leq \frac{1}{1-2(4+\sqrt{2})\,c}\left(D^{\beta}_{A}(\rho||\sigma) + D^{\beta}_{B}(\rho||\sigma)\right), \tag{3}$$

for every $\sigma = \mathcal{E}^*_{AB}(\sigma)$.



Geometric recursive argument, Bardet-C-Rouzé '19

$$\alpha\left(\mathcal{L}_{\Lambda}^{\beta *}\right) \geq \Psi(L_0) \min_{R \in \mathcal{R}_{L_0}} \alpha_{\Lambda}\left(\mathcal{L}_{R}^{\beta^*}\right),$$

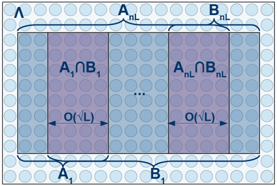
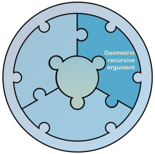


Figure: Splitting in A_n and B_n.

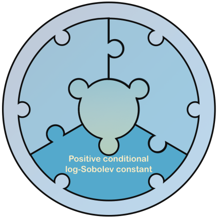


Theorem, Junge-LaRacuente-Rouzé '19

Given $\Lambda \subset\subset \mathbb{Z}^d$, $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ the Lindbladian associated to the Davies dynamics and a finite lattice and $A \subset \Lambda$, we have

$$\alpha_\Lambda \left(\mathcal{L}_A^{\beta *} \right) \geq \psi(|A|) > 0,$$

where $\psi(|A|)$ might depend on Λ, but is independent of its size.



Open problems

Problem 1

Can we use any of the quasi-factorization results to prove log-Sobolev constants in a more general setting?

Problem 2

Does the heat-bath example hold for greater dimension?

Problem 3

Is there a better definition for conditional relative entropy?


thank you
danke
рахмат
謝謝
ngiyabonga
شكراً جزيلاً
Баярлалаа
спасибо
teşekkür ederim
gracias
merci
obrigado
dziękuję
bedankt
go raibh maith agat
sukriya
kop khun krap
terima kasih
grazie
arigatô
dakujem
mochchakkeram
tapadh leat
dank je
мерси
ありがとう
감사합니다
ευχαριστώ
hvala
mauruuru
sagolun
takk
taiku
хвала
asante
manana
obrigada
tenki
murakoze
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shukriya
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rahmat
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bayarlalaa
gràcie
köszönöm
nanni
nandri
kiitos
dankie
dhanyavad
faafetai lava
vinaka
спасибі
blagodaram
mersi
kia ora
barka
welalin
tack
spas
misaotra
matondo
paldies
grazzi
mahalo
djiere dieuf
tau
akun
dankon
aciū