# Superadditivity of Quantum Relative Entropy for General States


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## Objective


The property of superadditivity of the quantum relative entropy states that, in a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, for every density operator $\rho_{AB}$ one has $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$. In this work, we provide an extension of this inequality for arbitrary density operators $\sigma_{AB}$.


## Introduction

- In a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, the **quantum relative entropy** of two states $\rho_{AB}$ and $\sigma_{AB}$ is given by:

$$D(\rho_{AB}||\sigma_{AB}) = \operatorname{tr}[\rho_{AB}(\log \rho_{AB} - \log \sigma_{AB})]$$

if $\operatorname{supp}(\rho_{AB}) \subseteq \operatorname{supp}(\sigma_{AB})$ and $+\infty$ otherwise.

- The property of **superadditivity** states that

$$D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

- As a consequence of **monotonicity**, the following holds for all states $\rho_{AB}$ and $\sigma_{AB}$:

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

## Main result

**Theorem**

For any bipartite states $\rho_{AB}, \sigma_{AB}$:

$$(1 + 2H(\sigma_{AB})_\infty) D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B),$$

where

$$H(\sigma_{AB}) = \sigma_A^{-1/2} \otimes \sigma_B^{-1/2}\, \sigma_{AB}\, \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB},$$

and $\mathbb{1}_{AB}$ denotes the identity operator in $\mathcal{H}_{AB}$.
Note that $H(\sigma_{AB}) = 0$ if $\sigma_{AB} = \sigma_A \otimes \sigma_B$.

## Steps 1 and 2 of the proof

**Step 1**

$$D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B) - \log \operatorname{tr} M, \tag{1}$$

where $M = \exp\left[\log \sigma_{AB} - \log \sigma_A \otimes \sigma_B + \log \rho_A \otimes \rho_B\right]$.

It holds that:

$$\begin{aligned} D(\rho_{AB}||\sigma_{AB}) &- [D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)] = \\ &= \operatorname{tr}\left[\rho_{AB}\left(\log \rho_{AB} - \underbrace{(\log \sigma_{AB} - \log \sigma_A \otimes \sigma_B + \log \rho_A \otimes \rho_B)}_{\log M}\right)\right] \\ &= D(\rho_{AB}||M) \geq -\log \operatorname{tr} M. \end{aligned}$$

**Step 2**

$$\log \operatorname{tr} M \leq \operatorname{tr}[L(\sigma_{AB})\,(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)], \tag{2}$$

where

$$L(\sigma_{AB}) = \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}.$$

**Theorem (Lieb):** Let $g$ a positive operator, and define

$$\mathcal{T}_g(f) = \int_0^\infty \mathrm{d}t\, (g+t)^{-1} f (g+t)^{-1}.$$

$\mathcal{T}_g$ is positive-semidefinite if $g$ is. We have that

$$\operatorname{tr}[\exp(-f + g + h)] \leq \operatorname{tr}\left[e^h \mathcal{T}_{e^f}(e^g)\right].$$

We apply Lieb's theorem to equation 1:

$$\begin{aligned} \operatorname{tr} M &\leq \operatorname{tr}[\rho_A \otimes \rho_B \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB})] \\ &= \operatorname{tr}\left[\rho_A \otimes \rho_B \underbrace{\left(\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}\right)}_{L(\sigma_{AB})}\right] + \underbrace{\operatorname{tr}[\rho_A \otimes \rho_B]}_{1}. \end{aligned}$$

By using the fact $\log(x) \leq x - 1$, we conclude

$$\log \operatorname{tr} M \leq \operatorname{tr} M - 1 \leq \operatorname{tr}[L(\sigma_{AB})\, \rho_A \otimes \rho_B].$$

**Lemma (Sutter et al.):** For $f \in \mathcal{S}_{AB}$ and $g \in \mathcal{A}_{AB}$ the following holds:

$$\mathcal{T}_g(f) = \int_{-\infty}^{\infty} dt\, \beta_0(t)\, g^{\frac{-1-it}{2}}\, f\, g^{\frac{-1+it}{2}},$$

with

$$\beta_0(t) = \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}.$$

**Lemma:** For every operator $O_A \in \mathcal{B}_A$ and $O_B \in \mathcal{B}_B$ the following holds:

$$\operatorname{tr}[L(\sigma_{AB})\, \sigma_A \otimes O_B] = \operatorname{tr}[L(\sigma_{AB})\, O_A \otimes \sigma_B] = 0.$$

## Steps 3 and 4 of the proof

**Step 3**

$$\operatorname{tr}[L(\sigma_{AB})\,(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)] \leq 2\left\|L(\sigma_{AB})\right\|_\infty D(\rho_{AB}||\sigma_{AB}). \tag{3}$$

In virtue of Hölder's inequality and tensorization of Schatten norms,

$$\begin{aligned} \operatorname{tr}[L(\sigma_{AB})\,(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)] &\leq \left\|L(\sigma_{AB})\right\|_\infty \left\|(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)\right\|_1 \\ &= \left\|L(\sigma_{AB})\right\|_\infty \left\|\rho_A - \sigma_A\right\|_1 \left\|\rho_B - \sigma_B\right\|_1. \end{aligned}$$

**Theorem (Pinsker):** For $\rho_{AB}$ and $\sigma_{AB}$ density matrices, it holds that

$$\left\|\rho_{AB} - \sigma_{AB}\right\|_1^2 \leq 2D(\rho_{AB}||\sigma_{AB}).$$

Using Pinsker's theorem and the data-processing inequality, we can conclude:

$$\operatorname{tr}[L(\sigma_{AB})\,(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)] \leq 2\left\|L(\sigma_{AB})\right\|_\infty D(\rho_{AB}||\sigma_{AB}).$$

**Step 4**

$$\left\|L(\sigma_{AB})\right\|_\infty \leq \left\|\sigma_A^{-1/2} \otimes \sigma_B^{-1/2}\, \sigma_{AB}\, \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB}\right\|_\infty. \tag{4}$$

## Conditional relative entropy

**Definition (Conditional relative entropy):**

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B).$$

**Proposition:** Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The following properties hold for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity.** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity.** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$, and if $D_A(\rho_{AB}||\sigma_{AB})=0$ then $\rho_{AB} = \mathbb{E}_A^*(\rho_{AB})$.
3. **Semi-additivity.** $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.
4. **Semi-superadditivity.** $D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$.

**Remark:** The *conditional relative entropy* extends the definition of *conditional entropy* for classical states.

## Quasi-factorization of the relative entropy

**Theorem**

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ be a tripartite Hilbert space and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$(1 - 2\left\|H(\sigma_{AC})\right\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \leq D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}), \tag{5}$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2}\, \sigma_{AC}\, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

## Log-Sobolev constant

**Theorem**

For $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ and $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$,

**Quasi-factorization $\Rightarrow$ Modified log-Sobolev constant.**

## References


- A. Capel, A. Lucia and D. Pérez-García, "Superadditivity of Quantum Relative Entropy for General States", *IEEE Transactions on Information Theory* (to appear), (2017) arxiv:1705.03521.
- A. Capel, A. Lucia and D. Pérez-García, "Quantum Conditional Relative Entropy and Quasi-Factorization of the Relative Entropy", in preparation.
- E.H. Lieb, "Convex trace functions and the Wigner–Yanase–Dyson conjecture", *Adv. Math.* **11**(3) (1973), 267-288, doi:10.1016/0001-8708(73)90011-X.
- M.S. Pinsker, *Information and Information Stability of Random Variables and Processes*, Holden Day (1964).
- D. Sutter, M. Berta and M. Tomamichel, "Multivariate Trace Inequalities", *Commun. Math. Phys.*, **352**(1) (2017), 37-58 doi:10.1007/s00220-016-2778-5.


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