Superadditivity of Quantum Relative Entropy for General States

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Objective

The property of superadditivity of the quantum relative entropy states that, in a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, for every density operator ρ_{AB} one has $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$. In this work, we provide an extension of this inequality for arbitrary density operators σ_{AB}.

Introduction

- In a bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, the **quantum relative entropy** of two states ρ_{AB} and σ_{AB} is given by:
$$D(\rho_{AB}||\sigma_{AB}) = \operatorname{tr}[\rho_{AB}(\log \rho_{AB} - \log \sigma_{AB})]$$
if $\operatorname{supp}(\rho_{AB}) \subseteq \operatorname{supp}(\sigma_{AB})$ and $+\infty$ otherwise.
- The property of **superadditivity** states that
$$D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$
- As a consequence of **monotonicity**, the following holds for all states ρ_{AB} and σ_{AB}:
$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

Main result

Theorem

For any bipartite states ρ_{AB}, σ_{AB}:
$$\left(1 + 2\left\|h\right\|_\infty\right) D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B),$$
where
$$h = \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB},$$
and
$$\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) = \int_0^\infty dt \, (\sigma_A \otimes \sigma_B + t\mathbb{1})^{-1} \sigma_{AB} \, (\sigma_A \otimes \sigma_B + t\mathbb{1})^{-1}.$$
Note that $h = 0$ if $\sigma_{AB} = \sigma_A \otimes \sigma_B$.

Step 1 of the proof

Step 1

For density matrices $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$, it holds that
$$D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B) - \log \operatorname{tr} M, \quad (1)$$
where $M = \exp\left[\log \sigma_{AB} - \log \sigma_A \otimes \sigma_B + \log \rho_A \otimes \rho_B\right]$ and equality holds (both sides being equal to zero) if $\rho_{AB} = \sigma_{AB}$.
Moreover, if $\sigma_{AB} = \sigma_A \otimes \sigma_B$, then $\log \operatorname{tr} M = 0$.

It holds that:
$$\begin{aligned} D(\rho_{AB}||\sigma_{AB}) &- [D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)] = \\ &= D(\rho_{AB}||\sigma_{AB}) - D(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) \\ &= \operatorname{tr}\left[\rho_{AB}\left(\log \rho_{AB} - \underbrace{(\log \sigma_{AB} - \log \sigma_A \otimes \sigma_B + \log \rho_A \otimes \rho_B)}_{\log M}\right)\right] \\ &= D(\rho_{AB}||M). \end{aligned}$$

Lemma: For $f, g \in \mathcal{A}^+$, the following holds:
$$D(f||g) \geq -\log \frac{\operatorname{tr}[g]}{\operatorname{tr}[f]}.$$

$$D(\rho_{AB}||\sigma_{AB}) - [D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)] \geq -\log \operatorname{tr} M.$$

Acknowledgments

Step 2 of the proof

Step 2

With the same notation of Step 1, we have that
$$\log \operatorname{tr} M \leq \operatorname{tr}(h \, \rho_A \otimes \rho_B), \quad (2)$$
where
$$h = \frac{1}{2}\left(\sigma_A^{-1} \otimes \sigma_B^{-1} \, \sigma_{AB} + \sigma_{AB} \, \sigma_A^{-1} \otimes \sigma_B^{-1}\right) - \mathbb{1}_{AB}.$$

Theorem (Lieb): Let g a positive operator, and define
$$\mathcal{T}_g(f) = \int_0^\infty dt \, (g+t)^{-1} f (g+t)^{-1}.$$
$\mathcal{T}_g$ is positive-semidefinite if g is. We have that
$$\operatorname{tr}[\exp(-f + g + h)] \leq \operatorname{tr}\left[e^h \mathcal{T}_{e^f}(e^g)\right].$$

We apply Lieb's theorem to equation 1:
$$\begin{aligned} \operatorname{tr} M &= \operatorname{tr}\left[\exp\left(\underbrace{\log \sigma_{AB}}_{g} - \underbrace{\log \sigma_A \otimes \sigma_B}_{f} + \underbrace{\log \rho_A \otimes \rho_B}_{h}\right)\right] \\ &\leq \operatorname{tr}[\rho_A \otimes \rho_B \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB})] \\ &= \operatorname{tr}\left[\rho_A \otimes \rho_B \underbrace{(\mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB})}_{h}\right] + \underbrace{\operatorname{tr}[\rho_A \otimes \rho_B]}_{1}. \end{aligned}$$

Finally, by using the fact $\log(x) \leq x - 1$, we conclude
$$\log \operatorname{tr} M \leq \operatorname{tr} M - 1 \leq \operatorname{tr}[h \, \rho_A \otimes \rho_B].$$

Step 3 of the proof

Step 3

With the same notation of the previous steps, the following holds:
$$\operatorname{tr}[h \, \rho_A \otimes \rho_B] \leq 2\left\|h\right\|_\infty D(\rho_{AB}||\sigma_{AB}).$$

Lemma (Sutter et al.): For $f \in \mathcal{S}_{AB}$ and $g \in \mathcal{A}_{AB}$ the following holds:
$$\mathcal{T}_g(f) = \int_{-\infty}^{\infty} dt \, \beta_0(t) \, g^{\frac{-1-it}{2}} \, f \, g^{\frac{-1+it}{2}},$$
with
$$\beta_0(t) = \frac{\pi}{2}(\cosh(\pi t) + 1)^{-1}.$$

Lemma: For every operator $O_A \in \mathcal{B}_A$ and $O_B \in \mathcal{B}_B$ the following holds:
$$\operatorname{tr}[h \, \sigma_A \otimes O_B] = \operatorname{tr}[h \, O_A \otimes \sigma_B] = 0.$$

This lemma implies that:
$$\operatorname{tr}[h \, \rho_A \otimes \rho_B] = \operatorname{tr}[h \, (\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)],$$
Thus,
$$\begin{aligned} \operatorname{tr}[h \, (\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)] &\leq \left\|h\right\|_\infty \left\|(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)\right\|_1 \\ &= \left\|h\right\|_\infty \left\|\rho_A - \sigma_A\right\|_1 \left\|\rho_B - \sigma_B\right\|_1. \end{aligned}$$

Theorem (Pinsker): For ρ_{AB} and σ_{AB} density matrices, it holds that
$$\left\|\rho_{AB} - \sigma_{AB}\right\|_1^2 \leq 2D(\rho_{AB}||\sigma_{AB}).$$

Using Pinsker's theorem and the data-processing inequality, we can conclude:
$$\operatorname{tr}[h \, \rho_A \otimes \rho_B] \leq 2\left\|h\right\|_\infty D(\rho_{AB}||\sigma_{AB}).$$

References

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