UNIVERSIDAD AUTONOMA DE MADRID

Máster en Matemáticas y Aplicaciones

Norm-attaining operators

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X^* dual of X, $x^* \in X^*$

$$\|x^*\| := \sup\{|x^*(x)| \,:\, x \in \mathbb{B}_X\}$$

x^* **attains its norm** when this supremum is a maximum, i.e.,

$$\exists x_0 \in \mathbb{S}_X \,:\, |x^*(x_0)| = \|x^*\|$$

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Reflexive space

X Banach space, X^{**} its bidual space

$$x \in X, \qquad J(x) : X^* \to \mathbb{K}$$

$$J(x)(f) = f(x) \qquad f \in X^*$$

A Banach space is **reflexive** when J is surjective.

James Theorem

A Banach space X is reflexive if, and only if, every continuous linear functional on X attains its maximum on $\mathbb{B}_X$.

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X, Y Banach spaces, $L(X,Y)$ set of bounded linear operators

$$\|T\| := \sup\{\|Tx\|_Y : x \in \mathbb{B}_X\} \qquad (T \in L(X,Y))$$

T **attains its norm** when this supremum is a maximum, i.e.,

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Problem (Bishop-Phelps)

$$¿\ \overline{NA(X,Y)} = L(X,Y)\ ?$$

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$$¿\ NA(X,Y) = L(X,Y)\ ?$$

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Objectives

1. Recopilation of results on norm-attaining operators
2. Study of endomorphisms
3. Elaboration of a monograph

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Problem $NA(X,Y) = L(X,Y)$

Problem

$$¿\ NA(X,Y) = L(X,Y)\ ?$$

Proposition

A Banach space X is reflexive if, and only if, for every finite dimensional Y, every $T \in L(X,Y)$ attains its norm.

X	Y	Result	Commentaries
Reflexive	$\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$)	James th.	
Reflexive	Finite dim.	Prop. 2.2.6	This result is a characterization of reflexivity : if it holds for every finite dim.Y, X is reflexive
Finite dim.	All	Prop. 2.2.8	Direct consequence of the compactness of the ball
Reflexive	?	Prop. 2.2.10	Necessary condition for X

X	Y	Result	Commentaries
$L^p(\mu)$, μ atomic and $1<p<\infty$	$L^r(\nu)$, ν atomic and $1<r<\infty$	Th. 2.2.17	This result is a characterization: If X and Y are of the type $C(S)$ or $L^p(\mu)$ and verify $(\sharp XY)$, they belong to one of these five cases
$L^p(\mu)$, $1<p<\infty$	$L^r(\nu)$, ν atomic and $1<r<2$		
$L^p(\mu)$, μ atomic and $2<p<\infty$	$L^r(\nu)$, $1<r<\infty$		
$L^p(\mu)$, $1<p<\infty$	$L^r(\nu)$, ν atomic and $r=1$		
$L^p(\mu)$, μ atomic and $2<p<\infty$	$L^r(\nu)$, $r=1$		

X	Result	Commentaries
$\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$)	James theorem	
Finite dimensional	Proposition 2.2.6	Among the classical Banach spaces, these are the only spaces verifying $(\sharp X)$

Problem $\overline{NA(X,Y)} = L(X,Y)$

Problem

¿ $\overline{NA(X,Y)} = L(X,Y)$?

The answer, in general, is negative.

Lindenstrauss' counterexample

$X = c_0$, Y strictly convex

$T \in NA(c_0, Y) \Rightarrow T \in F(c_0, Y)$

If there exists a non-compact operator from c_0 to Y, then $\overline{NA(c_0,Y)} \neq L(c_0,Y)$

Problem

¿ $\overline{NA(X)} = L(X)$?

If Y is strictly convex and isomorphic to c_0, $X = c_0 \oplus_\infty Y$

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Properties A and B

Properties A and B

X has **property** A if $\overline{NA(X,Y)} = L(X,Y) \quad \forall Y$

Y has **property** B if $\overline{NA(X,Y)} = L(X,Y) \quad \forall X$

Lindenstrauss-Zizler Theorem

Lind.: $\overline{\{T \in L(X,Y) \,:\, T^{**} \in NA(X^{**},Y^{**})\}} = L(X,Y) \quad \forall X,Y$

Zizler: $\overline{\{T \in L(X,Y) \,:\, T^{*} \in NA(Y^{*},X^{*})\}} = L(X,Y) \quad \forall X,Y$

$\Rightarrow$ Every reflexive Banach space has property A.

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$$\{(x_\lambda, x_\lambda^*) : \lambda \in \Lambda\} \subset \mathbb{S}_X \times \mathbb{S}_{X^*}, 0 \leq \rho < 1$$

(1) $x_\lambda^*(x_\lambda) = 1 \quad \forall \lambda \in \Lambda$

(2) $\lambda, \mu \in \Lambda,\ \lambda \neq \mu \Rightarrow |x_\lambda^*(x_\mu)| \leq \rho$

(3α) $\|x^*\| = \sup\{|x^*(x_\lambda)| : \lambda \in \Lambda\} \quad \forall x^* \in X^*$ (ej: ℓ_1)

(3β) $\|x\| = \sup\{|x_\lambda^*(x)| : \lambda \in \Lambda\} \quad \forall x \in X$ (ej: c_0, ℓ_∞)

Lindenstrauss

$$\alpha \Rightarrow A$$

$$\beta \Rightarrow B$$

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Partington Theorem

Every Banach space can be renormed with β.

Schachermayer Theorem

Every WCG Banach space can be renormed with α.

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Schachermayer Theorem

Every WCG Banach space can be renormed with α.

Relation with the Radon-Nikodym property

Dentability

X Banach space, C subset of X,
C is **dentable** if, for every $\varepsilon > 0$, we can find $x \in C$ such that $x \notin \overline{\text{co}}(C \setminus (x + \varepsilon \mathbb{B}_X))$.

Radon-Nikodym property

A Banach space X **has the RNP** if, and only if, every bounded subset of X is dentable.

Bourgain Theorem

$$RNP \Rightarrow A \text{ (for every equivalent norm)}$$

Huff Theorem

$$X \text{ no } RNP \Rightarrow \exists X_1 \sim X \sim X_2 : \overline{NA(X_1, X_2)} \neq L(X_1, X_2)$$

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Non-linear optimization principle of Bourgain-Stegall

$RNP \Leftarrow A$ (for every equivalent norm)

Conjecture

¿ RNP $\Leftrightarrow \overline{NA(X)} = L(X)$ for every equivalent norm ?

Proposition

Y Banach space, $X \cong Y \oplus Y$

$X \cong Y \oplus_1 Y \;\Rightarrow\; \|x\|_X = \|y_1\|_Y + \|y_2\|_Y \;\; \forall x = (y_1, y_2)$

$X \cong Y \oplus_\infty Y \;\Rightarrow\; \|x\|_X = \max\{\|y_1\|_Y, \|y_2\|_Y\} \;\; \forall x = (y_1, y_2)$

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Counterexamples

Gowers' counterexample

No infinite dimensional Hilbert space has property B.

For $1 < p < \infty$, ℓ_p and L_p do not have property B.

Acosta's counterexample

No infinite dimensional strictly convex Banach space has property B.

In another result, ℓ_1 and L_1 do not have property B.

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Open problems

- Do finite dimensional spaces have property B?
  In particular, does $\mathbb{R}^2$, with the euclidean norm, have property B?
- Characterize the compacts K such that $C(K)$ has property B.

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