Weak Quasi-Factorization for the Belavkin-Staszewski Relative Entropy

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Belavkin-Staszewski relative entropy

(Umegaki) relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space and ρ_{AB}, σ_{AB} two positive states on it. Their **Umegaki relative entropy** is given by:

$$D(\rho_{AB} \| \sigma_{AB}) := \operatorname{tr}[\rho_{AB}(\log \rho_{AB} - \log \sigma_{AB})] .$$

Belavkin-Staszewski relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a bipartite Hilbert space and ρ_{AB}, σ_{AB} two positive states on it. Their **Belavkin-Staszewski relative entropy** (BS-entropy for short) is given by:

$$\widehat{D}(\rho_{AB} \| \sigma_{AB}) := \operatorname{tr}\left[\rho_{AB} \log\left(\rho_{AB}^{1/2} \sigma_{AB}^{-1} \rho_{AB}^{1/2}\right)\right] .$$

Relation

The following holds for any positive states ρ_{AB}, σ_{AB}:

$$D(\rho_{AB} \| \sigma_{AB}) \leq \widehat{D}(\rho_{AB} \| \sigma_{AB}) ,$$

and the inequality is strict if, and only if, $[\rho_{AB}, \sigma_{AB}] \neq 0$.

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Motivation: Quasi-factorization relative entropy

Conditional relative entropy (C.-Lucia-Pérez García '18)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. We define the **conditional relative entropy** in A by

$$D_A(\rho_{AB}||\sigma_{AB}) := D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B) \, .$$

Given a positive state σ_{AB}, denote:

$$H(\sigma_{AB}) := \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \, .$$

Quasi-factorization relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following inequality holds whenever $\left\| H(\sigma_{AB}) \right\|_\infty < 1/2$:

$$D(\rho_{AB}||\sigma_{AB}) \leq \frac{1}{1 - 2\left\| H(\sigma_{AB}) \right\|_\infty} \left[D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}) \right] \, ,$$

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Applications: QF relative entropy

General superadditivity relative entropy:

The previous result is equivalent to (C.-Lucia-Pérez García '18):

$$\boxed{(1+2\|H(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

Due to:

- **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

we have

$$2D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B).$$

Modified logarithmic Sobolev inequalities:

- Quasi-factorization type inequalities are the key tool to prove positivity of **MLSI** for quantum many-body systems:

$$\alpha\, D(\rho_{AB}||\sigma_{AB}) \leq -\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_{AB})(\log \rho_{AB} - \log \sigma_{AB})]\,,$$

for $\mathcal{L}_\Lambda^*$ the generator of a quantum Markov semigroup.

Weak quasi-factorization for the BS-entropy

BS-entropy:

$$\widehat{D}(\rho_{AB}\|\sigma_{AB}) := \operatorname{tr}\left[\rho_{AB}\log\left(\rho_{AB}^{1/2}\sigma_{AB}^{-1}\rho_{AB}^{1/2}\right)\right].$$

Conditional BS-entropy (Bluhm-C.-Pérez Hernández '21)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. We define the **conditional BS-entropy** in A by:

$$\widehat{D}_A(\rho_{AB}\|\sigma_{AB}) := \widehat{D}(\rho_{AB}\|\sigma_{AB}) - \widehat{D}(\rho_B\|\sigma_B).$$

Weak quasi-factorization BS-entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. A result of **weak quasi-factorization of the BS-entropy** is an inequality of the form:

$$\widehat{D}(\rho_{AB}\|\sigma_{AB}) \leq M\left[\widehat{D}_A(\rho_{AB}\|\sigma_{AB}) + \widehat{D}_B(\rho_{AB}\|\sigma_{AB})\right] + L,$$

for $M \geq 1$ and $L \geq 0$.

Weak quasi-factorization for the BS-entropy

For σ_{AB} a positive state on $\mathcal{H}_{AB}$, denote:

$$H(\sigma_{AB}) := \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \, .$$

Weak QF BS-entropy I (Bluhm-C.-Pérez Hernández '21)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following inequality holds whenever $\left\| H(\sigma_{AB}) \right\|_\infty < 1/2$:

$$\widehat{D}(\rho_{AB} || \sigma_{AB}) \leq \widetilde{M}(\sigma_{AB}) \left[\widehat{D}_A(\rho_{AB} || \sigma_{AB}) + \widehat{D}_B(\rho_{AB} || \sigma_{AB}) \right] + \widetilde{L}(\rho_{AB}, \sigma_{AB}),$$

where

$$\widetilde{M}(\sigma_{AB}) := \frac{1}{1 - 2\left\| H(\sigma_{AB}) \right\|_\infty} \, ,$$

and

$$\widetilde{L}(\rho_{AB}, \sigma_{AB}) \leq f \left(\left\| \left[\rho_A^{1/2}, \sigma_A^{-1/2} \right] \right\|_\infty , \left\| \left[\rho_B^{1/2}, \sigma_B^{-1/2} \right] \right\|_\infty \right) .$$

- If $\sigma_{AB} = \sigma_A \otimes \sigma_B$, then $\widetilde{M}(\sigma_{AB}) = 1$.
- If $\rho_A^{1/2} \sigma_A^{-1/2}$ and $\rho_B^{1/2} \sigma_B^{-1/2}$ are normal (in particular, if $[\rho_A, \sigma_A] = [\rho_B, \sigma_B] = 0$), then $\widetilde{L}(\rho_{AB}, \sigma_{AB}) = 0$.

Weak quasi-factorization for the BS-entropy

Weak QF BS-entropy II (Bluhm-C.-Pérez Hernández '21)

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following inequality holds whenever $\left\| \sigma_{AB} - \sigma_A \otimes \sigma_B \right\|_\infty \sigma_{\min}^{-2} < d_A d_B / 2$:

$$\widehat{D}(\rho_{AB}||\sigma_{AB}) \leq M(\sigma_{AB}) \Big[\widehat{D}_A(\rho_{AB}||\sigma_{AB}) + \widehat{D}_B(\rho_{AB}||\sigma_{AB}) \Big] + L(\rho_{AB}, \sigma_{AB}),$$

where
$$M(\sigma_{AB}) := \frac{1}{1 - \frac{2\,\sigma_{\min}^{-2}}{d_A d_B} \left\| \sigma_{AB} - \sigma_A \otimes \sigma_B \right\|_\infty},$$

for $\sigma_{\min}$ the minimal eigenvalue of σ_{AB}, d_A and d_B the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and

$$L(\rho_{AB}, \sigma_{AB}) := M(\sigma_{AB}) \left(\left\langle \sigma_A \otimes \sigma_B, \sigma_A^{-1} \otimes \sigma_B^{-1} \right\rangle_{\rho_A \otimes \rho_B} - 1 \right) .$$

- If $\sigma_{AB} = \sigma_A \otimes \sigma_B$, then $M(\sigma_{AB}) = 1$.
- If $\rho_A^{1/2} \sigma_A^{-1/2}$ and $\rho_B^{1/2} \sigma_B^{-1/2}$ are normal (in particular, if $[\rho_A, \sigma_A] = [\rho_B, \sigma_B] = 0$), then $L(\rho_{AB}, \sigma_{AB}) = 0$.

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Equivalence to non-superadditivity

If $L(\rho_{AB}, \sigma_{AB}) = 0$ (or $\widetilde{L}(\rho_{AB}, \sigma_{AB}) = 0$), the previous results would be equivalent to **superadditivity** for the BS-entropy:

$$(1 + M(\sigma_{AB}))\widehat{D}(\rho_{AB}||\sigma_{AB}) \geq \widehat{D}(\rho_A||\sigma_A) + \widehat{D}(\rho_B||\sigma_B)\,,$$

and, since $M(\sigma_{AB}) = 1$ for $\sigma_{AB} = \sigma_A \otimes \sigma_B$,

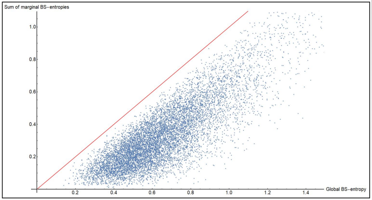
$$\widehat{D}(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq \widehat{D}(\rho_A||\sigma_A) + \widehat{D}(\rho_B||\sigma_B)\,.$$

However, continuity, additivity, superadditivity and monotonicity characterize the **relative entropy** (Wilming et at. '17, Matsumoto '10).

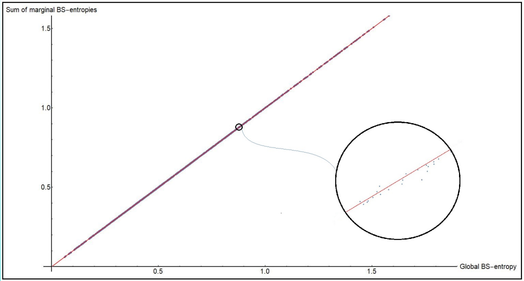
Therefore, the **additive term** is necessary in the quasi-factorization for the BS-entropy.

Numerical evidence of non-superadditivity

We plot $\widehat{D}(\rho_{AB}||\sigma_A \otimes \sigma_B)$ against $\widehat{D}(\rho_A||\sigma_A) + \widehat{D}(\rho_B||\sigma_B)$ for 10000 pairs of random 2×2 positive definite matrices ρ_{AB} and σ_{AB}:



Now we assume $\rho_{AB} := \dfrac{\eta_A \otimes \eta_B + \varepsilon\lambda_{AB}}{\operatorname{tr}[\eta_A \otimes \eta_B + \varepsilon\lambda_{AB}]}$.



Summary and possible applications

Summary:

- We have proven results of **weak quasi-factorization for the BS-entropy**:

$$\widehat{D}(\rho_{AB}||\sigma_{AB}) \leq M \left[\widehat{D}_A(\rho_{AB}||\sigma_{AB}) + \widehat{D}_B(\rho_{AB}||\sigma_{AB}) \right] + L \, .$$

- Our results are equivalent to a violation of the property of **superadditivity** for the BS-entropy.

Possible applications:

- In **cryptography**, for entropy accumulation purposes.
- In **many-body systems**, to prove positivity of modified logarithmic Sobolev inequalities.

For further information, see **arXiv:2101.10312**.

Thank you for your attention!