When does a quantum many-body system mix rapidly?

Quantum logarithmic Sobolev inequalities via quasi-factorization of the relative entropy

Ángela Capel Cuevas
(Universität Tübingen)

Mathematics Colloquium, Universität Tübingen, 27 June 2022

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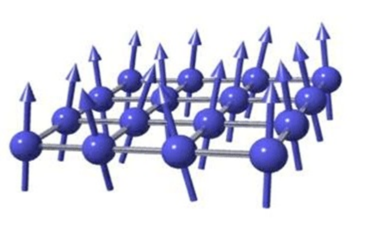

Notation: A (closed) quantum many-body system

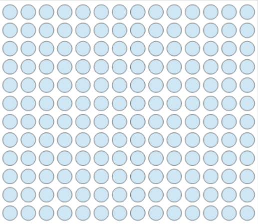

Figure: A quantum spin lattice system.

- Finite lattice $\Lambda \subset\subset \mathbb{Z}^d$.
- To every site $x \in \Lambda$ we associate $\mathcal{H}_x$ $(= \mathbb{C}^D)$.
- The global Hilbert space associated to Λ is $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$.
- The set of bounded linear endomorphisms on $\mathcal{H}_\Lambda$ is denoted by $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$.
- The set of density matrices is denoted by $\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{\rho_\Lambda \in \mathcal{B}_\Lambda \, : \, \rho_\Lambda \geq 0 \text{ and } \operatorname{tr}[\rho_\Lambda] = 1\}$.

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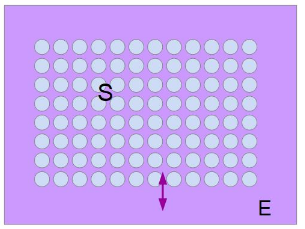
No experiment can be executed at zero temperature or be completely shielded from noise.



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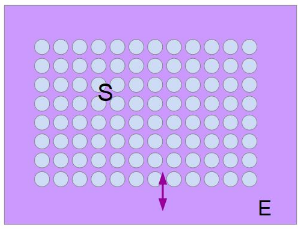

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Postulates of quantum mechanics

Postulate 1

Given an isolated physical system, there is a complex Hilbert space $\mathcal{H}$ associated to it, which is known as the **state space** of the system.

Moreover, the physical system is completely described by its **state vector**, which is a unitary vector in the state space.

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Given an isolated physical system, its evolution is described by a **unitary transformation** in the Hilbert space.

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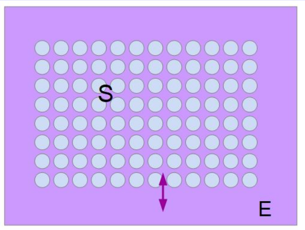
Evolution of an (open) quantum many-body system.



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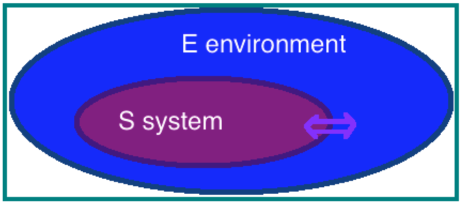

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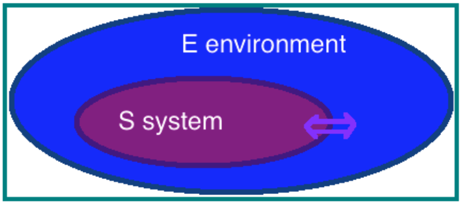

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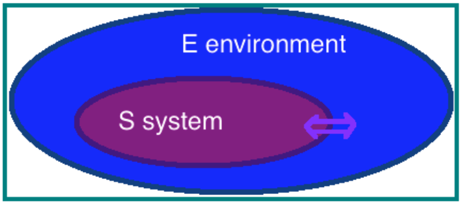

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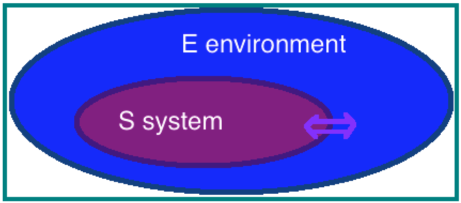

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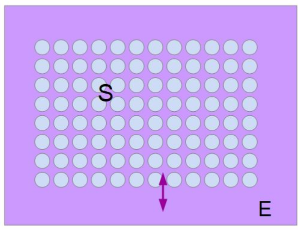

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Continuous-time description: For every $t \geq 0$, the corresponding time slice is a realizable evolution $\mathcal{T}_t$ (quantum channel).

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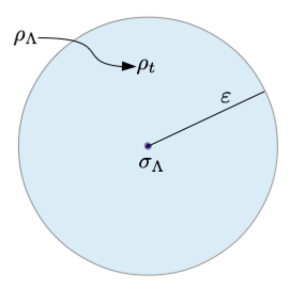

Dissipative quantum systems

Mixing $\Leftrightarrow$ Convergence

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for every $f, g \in \mathcal{B}_\Lambda$ and Hermitian, where

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Mixing time

- Under the previous conditions, there is always convergence to σ_Λ.
- How fast does convergence happen?

Note $\mathcal{T}_\infty^*(\rho) := \sigma_\Lambda$ for every ρ.

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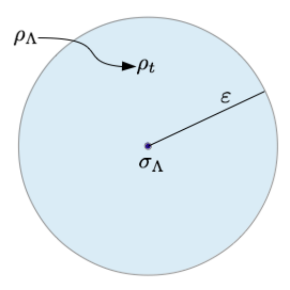

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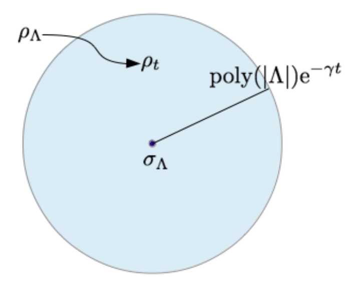

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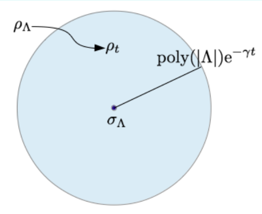

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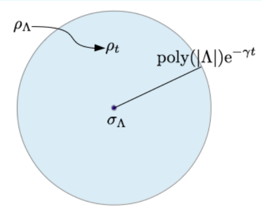

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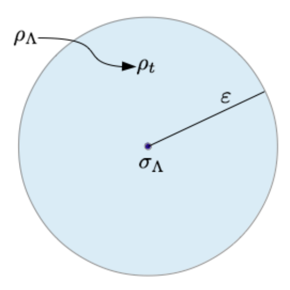

Quantum dissipative evolutions useful?

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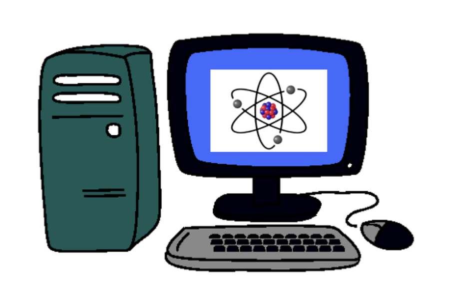

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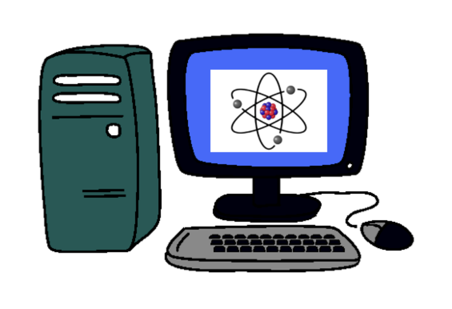

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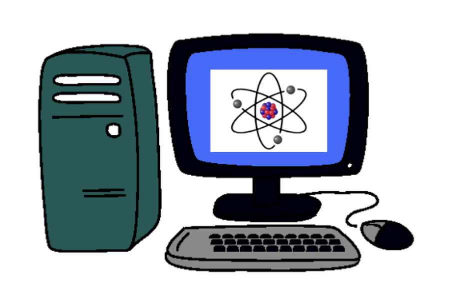

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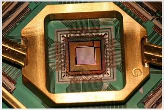

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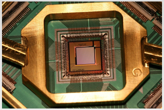

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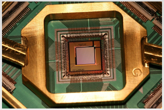

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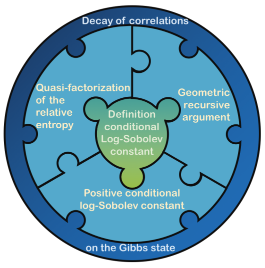

Modified logarithmic Sobolev inequality (MLSI)

(in this talk, we simply call it **log-Sobolev inequality**)

Recall: $\rho_t := \mathcal{T}_t^*(\rho)$.

Master equation:

$$\partial_t \rho_t = \mathcal{L}_\Lambda^*(\rho_t).$$

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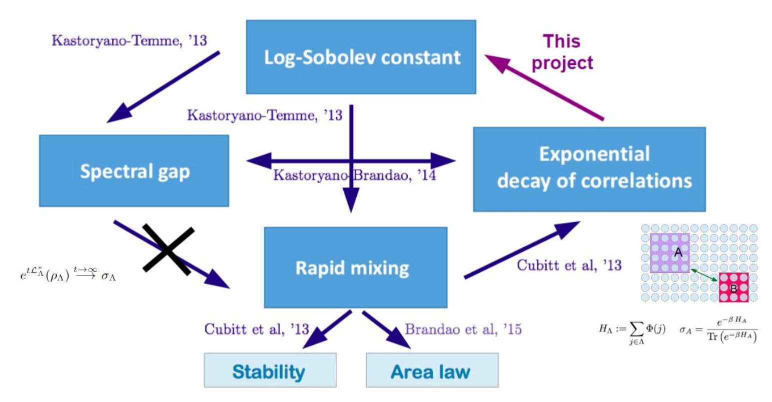
When does a dissipative quantum many-body system mix rapidly?

When does a dissipative quantum many-body system mix rapidly?

ANSWER: When our dissipative quantum many-body system has a **positive MLSI**, it mixes rapidly.

Remark: And when it only has a positive spectral gap, in general it does not mix rapidly.

Quantum spin systems



Exp. decay of correlations:

$$\sup_{\|O_A\|=\|O_B\|=1} |\mathrm{tr}[O_A \otimes O_B(\sigma_{AB} - \sigma_A \otimes \sigma_B)]| \leq K\, \mathrm{e}^{-\gamma d(A,B)} \quad .$$

Decay of correlations on Gibbs state

Motivation

Describe the **correlation properties** of **Gibbs states** of local Hamiltonians.

- **Hamiltonian:** $H_\Lambda = H_A + H_B + H_{(A\cup B)^c} + H_{\partial A} + H_{\partial B}$,
- **Gibbs state:** $\sigma_\Lambda(\beta) = \mathrm{e}^{-\beta H_\Lambda} \, / \, \mathrm{Tr}[\mathrm{e}^{-\beta H_\Lambda}]$.

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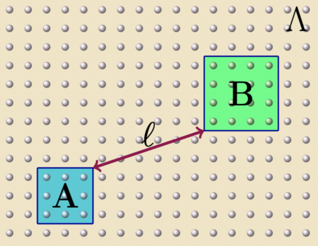

$\ell := \mathrm{dist}(A, B)$

Questions:

For non-commuting Hamiltonians:

$$\mathrm{e}^{-\beta H_{A\cup B}} \approx \mathrm{e}^{-\beta H_A}\,\mathrm{e}^{-\beta H_B} \ ?$$

$$\mathrm{tr}_{A^c}[\sigma_\Lambda] \otimes \mathrm{tr}_{B^c}[\sigma_\Lambda] := (\sigma_\Lambda)_A \otimes (\sigma_\Lambda)_B \approx$$
$$\mathrm{tr}_{(A\cup B)^c}[\sigma_\Lambda] := (\sigma_\Lambda)_{A\cup B} \ ?$$

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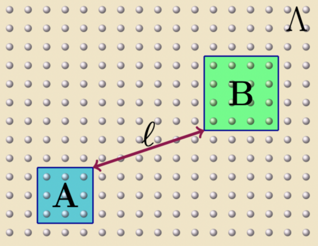

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Decay of correlations on Gibbs state

3 different forms of **decay of correlations**.

Operator correlation

$$\text{Corr}_\sigma(A:B) := \sup_{\|O_A\|=\|O_B\|=1} |\text{tr}[O_A \otimes O_B(\sigma_{AB} - \sigma_A \otimes \sigma_B)]|$$

Mutual information

$$I_\sigma(A:B) := D(\rho_{AB}||\rho_A \otimes \rho_B)$$

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Ángela Capel Cuevas (Universität Tübingen)

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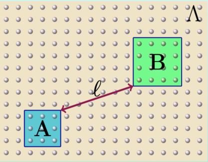
Mixing condition

$$\|h(\sigma_{AB})\|_\infty = \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty$$



Relation:

$$\frac{1}{2}\mathrm{Corr}_\sigma(A:B)^2 \leq I_\sigma(A:B)$$

$$\leq \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty .$$

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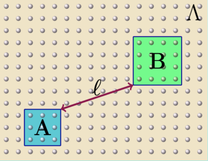

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Quantum spin systems

Thermalization

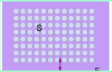
MLSI (log-Sobolev)

$D(\rho_t||\sigma_\Lambda) \leq D(\rho_\Lambda||\sigma_\Lambda) e^{-2\,\alpha(\mathcal{L}_\Lambda^*)\,t}$

Rapid Mixing

$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}$

$e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$



Spectral gap

$\text{Var}(\rho_t, \sigma_\Lambda) \leq \text{Var}(\rho_\Lambda, \sigma_\Lambda) e^{-\lambda(\mathcal{L}_\Lambda^*)\,t}$

Decay of correlations

Mixing condition

$\left\| \sigma_A^{-1} \otimes \sigma_B^{-1} \sigma_{AB} - \mathbb{1}_{AB} \right\|_\infty \leq K e^{-\gamma\, d(A,B)}$

Mutual information

$I_\sigma(A:B) \leq K e^{-\gamma\, d(A,B)}$

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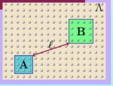
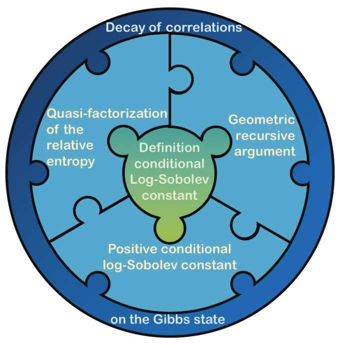


When does a dissipative quantum many-body system mix rapidly?

When does a dissipative quantum many-body system have a **positive MLSI**?



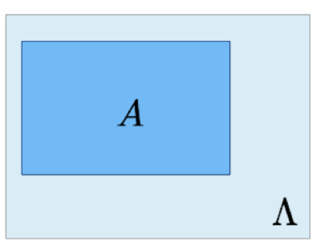
Objective

MLSI constant

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

What do we want to prove?

$$\liminf_{\Lambda \nearrow \mathbb{Z}^d} \alpha(\mathcal{L}_\Lambda^*) \geq \Psi(|\Lambda|) > 0.$$



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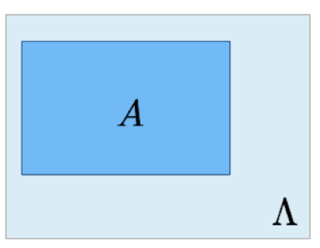

Can we prove something like

$$\alpha(\mathcal{L}_\Lambda^*) \geq \Psi(|A|)\, \alpha(\mathcal{L}_A^*) > 0 \ ?$$

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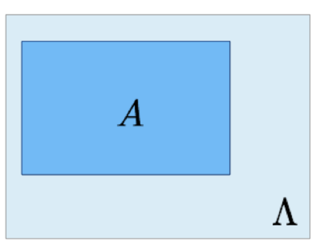

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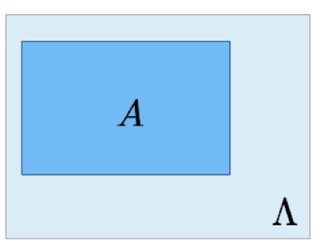

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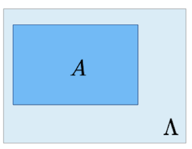
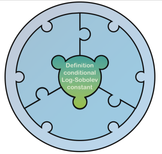
Conditional MLSI constant





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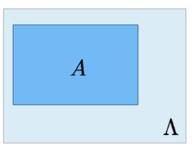
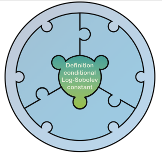
Conditional MLSI constant





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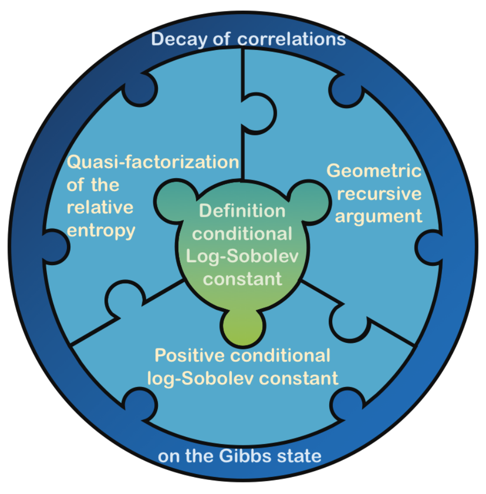
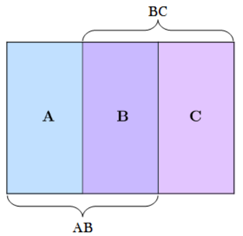
Strategy

Used in (C.-Lucia-Pérez García ’18) and (Bardet-C.-Lucia-Pérez García-Rouzé, ’19).



Quasi-factorization of the relative entropy



Quasi-factorization of the relative entropy

Given $\Lambda = ABC$, it is an inequality of the form:

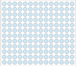
$$D(\rho_\Lambda \| \sigma_\Lambda) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_\Lambda \| \sigma_\Lambda) + D_{BC}(\rho_\Lambda \| \sigma_\Lambda) \right] ,$$

for $\rho_\Lambda, \sigma_\Lambda \in \mathcal{D}(\mathcal{H}_{ABC})$, where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

Example: Tensor product fixed point

(C.-Lucia-Pérez García '18) $\mathcal{L}^*_\Lambda(\rho_\Lambda) = \sum_{x\in\Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda)$ **heat-bath**

$$D_x(\rho_\Lambda||\sigma_\Lambda) := D(\rho_\Lambda||\sigma_\Lambda) - D(\rho_{x^c}||\sigma_{x^c})$$



$$\sigma_\Lambda = \bigotimes_{x\in\Lambda} \sigma_x \,,$$

$$D(\rho_\Lambda||\sigma_\Lambda) \leq$$

$$\leq \sum_{x\in\Lambda} D_x(\rho_\Lambda||\sigma_\Lambda)$$

$\alpha_\Lambda(\mathcal{L}^*_x) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}^*_x(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]}{2D_x(\rho_\Lambda||\sigma_\Lambda)}$

$$\leq \sum_{x\in\Lambda} \frac{-\operatorname{tr}[\mathcal{L}^*_x(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}^*_x)}$$

$$\leq \frac{1}{2\inf\limits_{x\in\Lambda} \alpha_\Lambda(\mathcal{L}^*_x)} \sum_{x\in\Lambda} -\operatorname{tr}[\mathcal{L}^*_x(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]$$

$$= \frac{1}{2\inf\limits_{x\in\Lambda} \alpha_\Lambda(\mathcal{L}^*_x)} \left(-\operatorname{tr}[\mathcal{L}^*_\Lambda(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]\right)$$

$$\leq \left(-\operatorname{tr}[\mathcal{L}^*_\Lambda(\rho_\Lambda)(\log\rho_\Lambda - \log\sigma_\Lambda)]\right).$$

Dynamics

Let $\sigma_\Lambda = \frac{e^{-\beta H_\Lambda}}{\operatorname{tr}\left[e^{-\beta H_\Lambda}\right]}$ be the Gibbs state of finite-range, commuting Hamiltonian.

Heat-bath generator

The **heat-bath generator** is defined as:

$$\mathcal{L}_\Lambda^{H;*}(\rho_\Lambda) := \sum_{x \in \Lambda} \left(\sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} - \rho_\Lambda \right)$$

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The **Davies generator** is given by:

$$\mathcal{L}_\Lambda^D(X) := i[H_\Lambda, X] + \sum_{x\in\Lambda} \mathcal{L}_x^D(X) \, ,$$

where the $\mathcal{L}_x^D$ are defined in terms of the Fourier coefficients of the correlation functions in the bath and the ones of the system couplings.

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The **Schmidt generator** (Bravyi-Vyalyi '05) can be written as:

$$\mathcal{L}_\Lambda^S(X) = \sum_{x \in \Lambda} \left(E_x^S(X) - X \right) ,$$

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Previous results

Let us recall: For $\alpha(\mathcal{L}_\Lambda^*)$ a MLSI constant,

$$\|\rho_t - \sigma_\Lambda\|_1 \leq \sqrt{2\log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\, t}.$$

Using the spectral gap $\lambda(\mathcal{L}_\Lambda^*)$:

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Let $\mathcal{L}_\Lambda^{H,D;*}$ be the **heat-bath** or **Davies** generator in 1D. Then, $\mathcal{L}_\Lambda^{H,D;*}$ has a positive spectral gap that is independent of the system size, for every temperature.

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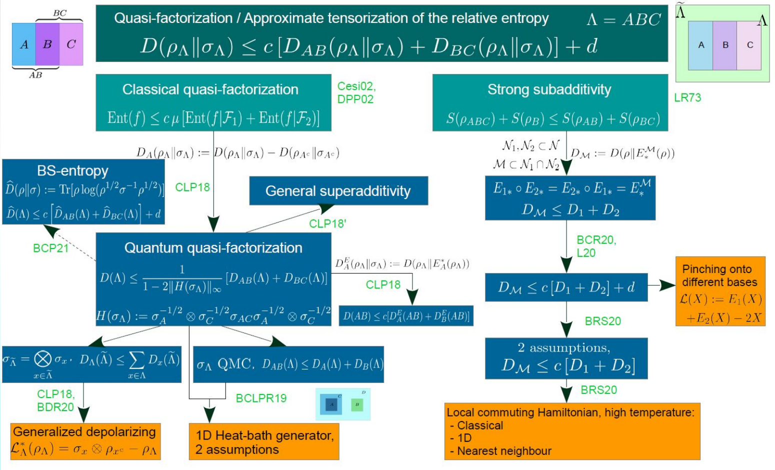

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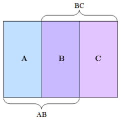

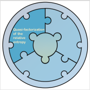

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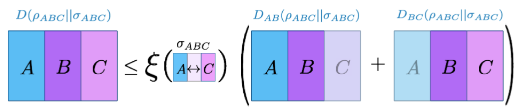

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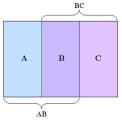

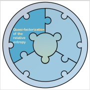

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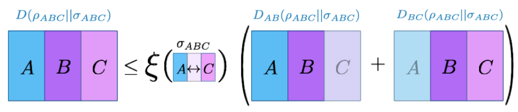
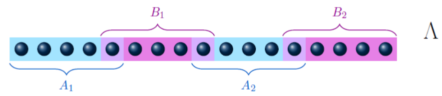
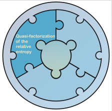


Sketch of the proof: Quasi-factorization





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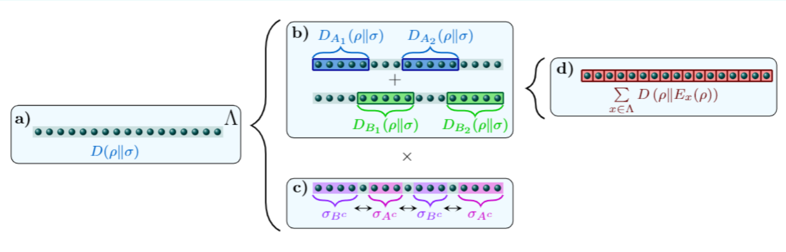

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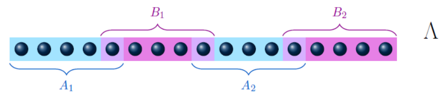

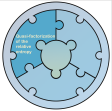

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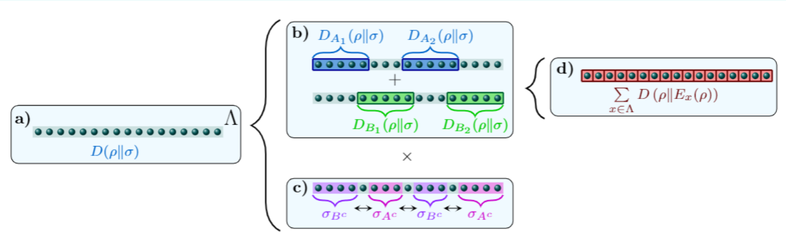

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Thank you for your attention!

Do you have any questions?

David Pérez-García
U. Complutense
Madrid

Angelo Lucia
U. Complutense
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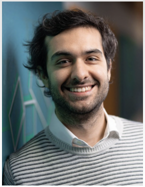
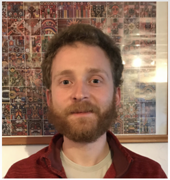


Cambyse Rouzé
T. U. Munich



Ivan Bardet
Inria Paris

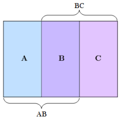
Daniel Stilck Franca
ENS Lyon

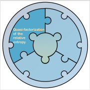
Antonio
Pérez-Hernández
UNED Madrid

Andreas Bluhm
U. Copenhagen

Li Gao
U. Houston

Proof: Conditional relative entropies + Quasi-factorization





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Let $\mathcal{H}_{ABC}$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. The following holds

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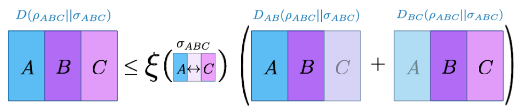
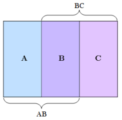
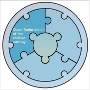


Proof: Conditional relative entropies + Quasi-factorization





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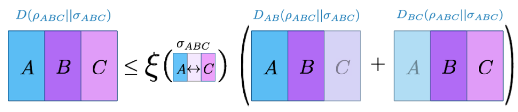
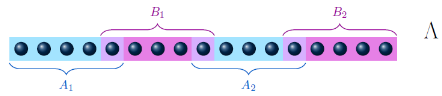
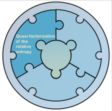


Proof: Quasi-factorization





$\sigma_\Lambda = \frac{\mathrm{e}^{-\beta H_\Lambda}}{\operatorname{tr}(\mathrm{e}^{-\beta H_\Lambda})}$ is the Gibbs state of a k-local, commuting Hamiltonian H_Λ.

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Quasi-factorization for quantum Markov chains

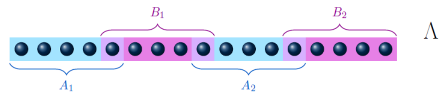
(Bardet-C.-Lucia-Pérez García-Rouzé'19)

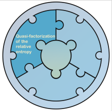
Since σ_Λ is a QMC between $A_i \leftrightarrow \partial(A_i) \leftrightarrow (A_i \cup \partial A_i)^c$, then:

$$D_A(\rho_\Lambda || \sigma_\Lambda) \leq \sum_i D_{A_i}(\rho_\Lambda || \sigma_\Lambda).$$

$$\sigma_\Lambda = \bigoplus_{j \in J} \sigma_{A_i (\partial a_i)_j^L} \otimes \sigma_{(\partial a_i)_j^R (A_i \cup \partial A_i)^c}$$

Proof: Quasi-factorization





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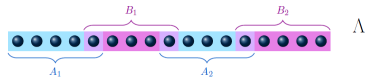
Proof: Decay of correlations



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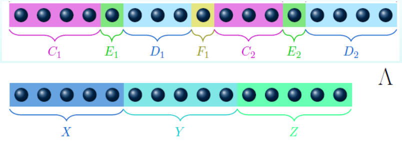

Decay of correlations, (Bluhm-C.-Pérez Hernández, '21)

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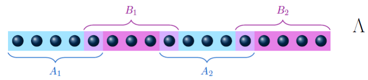

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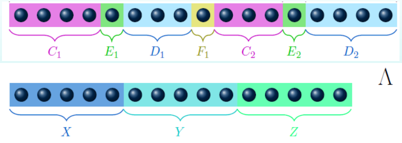

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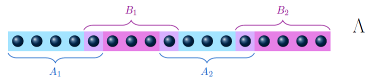

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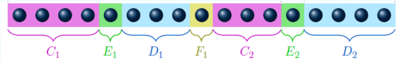

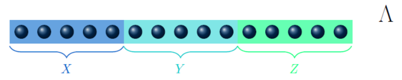

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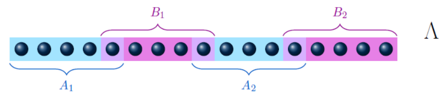
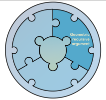
Proof: Geometric recursive argument





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Comparison conditional rel. ent. (Bardet-C.-Rouzé, '20)

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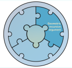
Therefore, by this and  +  , we have:

$$D(\rho_\Lambda \| \sigma_\Lambda) \leq \xi(\sigma_{A^c B^c}) \sum_i \left[D_{A_i}^E(\rho_\Lambda \| \sigma_\Lambda) + D_{B_i}^E(\rho_\Lambda \| \sigma_\Lambda) \right],$$

and thus $\alpha(\mathcal{L}_\Lambda^{H;*}) \geq \dfrac{K}{\xi(\sigma_{A^c B^c})} \min \left\{ \alpha_{A_i}(\mathcal{L}_\Lambda^{H;*}), \alpha_{B_i}(\mathcal{L}_\Lambda^{H;*}) \right\},$

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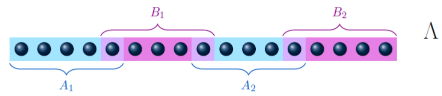

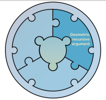

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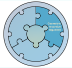
Therefore, by this and  +  , we have:

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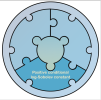
Proof: Positive CMLSI



Reduction of conditional relative entropies (Gao-Rouzé, '21)

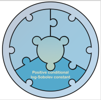
$$D(\rho_\Lambda \| E^*_{A_i}(\rho_\Lambda)) \leq 4 k_{A_i} \sum_{j \in A_i} D(\rho_\Lambda \| E^*_j(\rho_\Lambda))$$

Reduction from CMLSI to Gap

$$k_{A_i} \propto \frac{1}{\ln \lambda} \, ,$$

where $\lambda < 1$ is a constant related to the spectral gap by the detectability lemma.

Proof: Positive CMLSI



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As a consequence of the non-closure of the spectral gap proved for 1D commuting Gibbs samplers (Kastoryano-Brando '16), $k_{A_i} = \mathcal{O}(\ln |\Lambda|)$ for $A_i = \mathcal{O}(\ln |\Lambda|)$.

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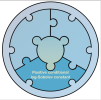

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CMLSI (Gao-Rouzé, '21)

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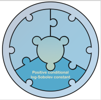

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Last step

Heat-bath cond. expectation:
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Davies and heat-bath dynamics (Bardet-C.-Rouzé, '20)

The conditional expectations associated to Davies and heat-bath dynamics coincide.

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Conclusion

For $\mathcal{L}_\Lambda^{D;*}$, there is a positive MLSI constant $\alpha(\mathcal{L}_\Lambda^{D;*}) = \Omega(\ln|\Lambda|^{-1})$. Therefore, $\mathcal{L}_\Lambda^{D;*}$ has rapid mixing.

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