A strengthened data processing inequality for the Belavkin-Staszewski relative entropy

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1. Motivation

2. Standard and maximal f-divergences

3. Main results
    - Equivalent conditions for equality on DPI
    - Strengthened DPI for the BS-entropy
    - Strengthened DPI for maximal f-divergences

Notation

- $\mathcal{H}$ finite-dimensional Hilbert space.
- $\mathcal{B}(\mathcal{H})$ algebra of bounded linear operators on it.
- $\mathcal{D}(\mathcal{H}) := \{\rho \in \mathcal{B}(\mathcal{H}) : \rho \geq 0, \operatorname{tr}[\rho] = 1\}$ density matrices.
  - Assume full rank.
- $\mathcal{M}$ matrix algebra and $\mathcal{N} \subset \mathcal{M}$ matrix subalgebra.
- $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ quantum channel.
- $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ conditional expectation.
  - $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$.
- $\Gamma := \sigma^{-1/2} \rho \sigma^{-1/2}$ and $\Gamma_{\mathcal{N}} := \sigma_{\mathcal{N}}^{-1/2} \rho_{\mathcal{N}} \sigma_{\mathcal{N}}^{-1/2}$.
- **Note:** Identify L_A, the left multiplication operator by A on $\mathcal{M}$, with A for $A \in \mathcal{M}$.

Main concepts

Relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **relative entropy** is defined as:

$$D(\sigma||\rho) := \operatorname{tr}[\sigma(\log \sigma - \log \rho)].$$

Belavkin-Staszewski relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **BS-entropy** is defined as:

$$D_{\mathrm{BS}}(\sigma||\rho) := \operatorname{tr}\left[\sigma \log \left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right].$$

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Relation between relative entropies

The following holds for every $\sigma > 0, \rho > 0$:

$$D_{\mathrm{BS}}(\sigma||\rho) \geq D(\sigma||\rho).$$

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Relative entropy of σ and ρ: $D(\sigma||\rho) := \mathrm{tr}[\sigma(\log \sigma - \log \rho)]$.

Quantum channel: $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ CPTP map.

- $\sigma > 0 \mapsto \mathcal{T}(\sigma) > 0$.
- $\mathcal{T} \otimes \mathrm{Id}_n : \mathcal{M} \otimes \mathcal{M}_n \to \mathcal{M} \otimes \mathcal{M}_n$ is positive for every $n \in \mathbb{N}$.
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$$D(\sigma||\rho) = D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \Leftrightarrow \sigma = \rho^{1/2}\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1/2}\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1/2}\right)\rho^{1/2}.$$

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Operational meaning of $D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho))$

- **Thermodynamics:** Cost of a certain quantum process.
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DPI for relative entropy: $D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \geq 0.$

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(Fawzi-Renner '15) $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, $\sigma_{ABC} > 0$ and $\rho_{ABC} = I_A \otimes \sigma_{BC}$, $\mathcal{T}(\cdot) = \operatorname{tr}_C[\cdot]$.

CMI: $I(A:C|B)_\sigma = D(\sigma_{ABC}||\rho_{ABC}) - D(\sigma_{BC}||\rho_{BC}).$

$$I(A:C|B)_\sigma \geq \inf_{\eta_{ABC}} \left(-2\log_2 F(\sigma_{ABC}, \eta_{ABC})\right),$$

where

$$F(\sigma_{ABC}, \eta_{ABC}) = \left\| \sqrt{\sigma_{ABC}}\sqrt{\eta_{ABC}} \right\|_1$$

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we have

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Extensions and improvements of the previous result:
$D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \geq (1), (2), (3)$, where:

$$(1) := -\int \beta_0(t) \log F\left(\sigma, \mathcal{R}_{\mathcal{T}}^{\rho,[t]} \circ \mathcal{T}(\sigma)\right) \mathrm{d}t \text{ (Junge et al. '15)},$$

with

$$\mathcal{R}_{\mathcal{T}}^{\rho,[t]}(\cdot) = \rho^{\frac{1+it}{2}} \mathcal{T}^* \left(\mathcal{T}(\rho)^{\frac{-1-it}{2}} (\cdot) \mathcal{T}(\rho)^{\frac{-1+it}{2}}\right) \rho^{\frac{1-it}{2}}$$

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Can we find a lower bound for the DPI in terms of $D(\sigma || \mathcal{R}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma))$*?*

Answer: It is not possible **(Brandao et al. '15, Fawzi[2] '17)**.

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Some definitions

Conditional expectation

Let $\mathcal{M}$ matrix algebra with matrix subalgebra $\mathcal{N}$. There exists a unique linear mapping $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ such that

1. $\mathcal{E}$ is a positive map,
2. $\mathcal{E}(B) = B$ for all $B \in \mathcal{N}$,
3. $\mathcal{E}(AB) = \mathcal{E}(A)B$ for all $A \in \mathcal{M}$ and all $B \in \mathcal{N}$,
4. $\mathcal{E}$ is trace preserving.

A map fulfilling (1)-(3) is called a *conditional expectation.*

Operator convex

Let $\mathcal{I} \subseteq \mathbb{R}$ interval and $f : \mathcal{I} \to \mathbb{R}$. If

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$

for all Hermitian A, $B \in \mathcal{B}(\mathcal{H})$ with spectrum contained in $\mathcal{I}$, all $\lambda \in [0,1]$, and for all finite-dimensional Hilbert spaces $\mathcal{H}$, then f is *operator convex.*

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Let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$S_f(\sigma \| \rho) = \operatorname{tr}\left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) \rho^{1/2}\right]$$

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Example: Let $f(x) = x \log x$. Then,

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defines the relative entropy $D(\sigma \| \rho)$.

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Let $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a TP map such that its dual map is a 2PTP map. Then, for every $\sigma > 0$, $\rho > 0$ on $\mathcal{M}$ and every operator convex function $f : (0, \infty) \to \mathbb{R}$,

$$S_f(\mathcal{T}(\sigma)\|\mathcal{T}((\rho)) \leq S_f(\sigma\|\rho).$$

Standard f-divergences

(Hiai-Mosonyi '17)

Standard f-divergences

Let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$S_f(\sigma \| \rho) = \operatorname{tr}\left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) \rho^{1/2}\right]$$

is the *standard f-divergence.*

Example: Let $f(x) = x \log x$. Then,

$$S_f(\sigma \| \rho) = \operatorname{tr}[\sigma(\log \sigma - \log \rho)]$$

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Standard f-divergences

Conditions for equality

Let $\sigma > 0$, $\rho > 0$ be on $\mathcal{M}$ and let $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a 2PTP linear map. Then, the following are equivalent:

1. There exists a TP map $\hat{\mathcal{T}} : \mathcal{B} \to \mathcal{M}$ such that $\hat{\mathcal{T}}(\mathcal{T}(\rho)) = \rho$ and $\hat{\mathcal{T}}(\mathcal{T}(\sigma)) = \sigma$.
2. $S_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) = S_f(\sigma \| \rho)$ for some operator convex function on $(0, \infty)$ such that $f(0^+) < \infty$ and
$$|\text{supp } \mu_f| \geq |\text{spec}(L_\sigma R_{\rho^{-1}}) \cup \text{spec}(L_{\Phi(\sigma)} R_{\Phi(\rho)^{-1}})|,$$
for a certain μ_f from the representation of the op-convex function.
3. $S_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) = S_f(\sigma \| \rho)$ for all operator convex f on $[0, \infty)$.
4. $\mathcal{R}^\rho_{\mathcal{T}}(\mathcal{T}(\sigma)) = \sigma$.

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is the *maximal f-divergence.*

Example: Let $f(x) = x \log x$. Then,

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Let $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a PTP linear map. Then,

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Conditions for equality

Let $\sigma > 0$, $\rho > 0$ be on $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a PTP linear map. Then, the following are equivalent:

1. $\hat{S}_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) = \hat{S}_f(\sigma \| \rho)$ for some non-linear operator convex function f on $[0, \infty)$.
2. $\hat{S}_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) = \hat{S}_f(\sigma \| \rho)$ for all operator convex functions f on $[0, \infty)$.
3. $\operatorname{tr}\left[\mathcal{T}(\sigma)^2 \mathcal{T}(\rho)^{-1}\right] = \operatorname{tr}\left[\sigma^2 \rho^{-1}\right]$.

Relation between f-divergences

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For every two states $\sigma > 0$, $\rho > 0$ on $\mathcal{M}$ and every operator convex function $f : (0, \infty) \to \mathbb{R}$,

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Remark: Difference

For maximal f-divergences, there is no equivalent condition for equality in DPI which provides a explicit expression of recovery for σ.

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Questions

BS recovery condition

Can we prove an equivalent condition for equality in DPI for the BS entropy (or for maximal f-divergences) which provides a explicit expression of recovery for σ?

Strengthened DPI for BS entropy

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Equivalent conditions for equality on DPI

$\Gamma := \sigma^{-1/2}\rho\sigma^{-1/2}$ and $\Gamma_{\mathcal{N}} := \sigma_{\mathcal{N}}^{-1/2}\rho_{\mathcal{N}}\sigma_{\mathcal{N}}^{-1/2}$
$\rho_{\mathcal{N}} := \mathcal{E}(\rho)$, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$

Equivalent conditions for equality on DPI (Bluhm-C. '19)

Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states on $\mathcal{M}$. Then, the following are equivalent:

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Consequences

Note: Although they can be seen as a consequence of the previous result, the following facts were previously known.

Corollary

$$\begin{aligned} \hat{S}_{\rm BS}(\sigma\|\rho) = \hat{S}_{\rm BS}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) &\Leftrightarrow \rho = \mathcal{T}_{\mathcal{E}}^{\sigma} \circ \mathcal{E}(\rho) \\ &\Leftrightarrow \sigma = \mathcal{T}_{\mathcal{E}}^{\rho} \circ \mathcal{E}(\sigma) \\ &\Leftrightarrow \hat{S}_{\rm BS}(\rho\|\sigma) = \hat{S}_{\rm BS}(\rho_{\mathcal{N}}\|\sigma_{\mathcal{N}}). \end{aligned}$$

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$$D(\sigma\|\rho) = D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \implies \hat{S}_{BS}(\sigma\|\rho) = \hat{S}_{BS}(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}).$$

Equivalently,

$$\sigma = \mathcal{R}^{\rho}_{\mathcal{E}} \circ \mathcal{E}(\sigma) \implies \sigma = \mathcal{T}^{\rho}_{\mathcal{E}} \circ \mathcal{E}(\sigma).$$

The converse of this result is false (**Jencová-Petz-Pitrik '09, Hiai-Mosonyi '17**).

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Strengthened DPI for the BS-entropy

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Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma > 0$, $\rho > 0$ be two quantum states onto $\mathcal{M}$. Then,

$$\hat{S}_{\mathrm{BS}}(\sigma \| \rho) - \hat{S}_{\mathrm{BS}}(\sigma_{\mathcal{N}} \| \rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8}\right)^4 \|\Gamma\|_\infty^{-4} \left\|\sigma^{-1}\right\|_\infty^{-2} \left\|\rho - \sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\right\|_2^4 .$$

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Let $\mathcal{M}$ be a matrix algebra with unital subalgebra $\mathcal{N}$. Let $\mathcal{E}:\mathcal{M}\to\mathcal{N}$ be the trace-preserving conditional expectation onto this subalgebra. Let $\sigma>0$, $\rho>0$ be two quantum states on $\mathcal{M}$ and let $f:(0,\infty)\to\mathbb{R}$ be an operator convex function with transpose $\tilde{f}$. We assume that $\tilde{f}$ is operator monotone decreasing and such that the measure $\mu_{-\tilde{f}}$ that appears in the representation of $-\tilde{f}$ is absolutely continuous with respect to Lebesgue measure. Moreover, we assume that for every $T\geq 1$, there exist constants $\alpha\geq 0$, $C>0$ satisfying $\mathrm{d}\mu_{-\tilde{f}}(t)/\mathrm{d}t\geq (CT^{2\alpha})^{-1}$ for all $t\in[1/T,T]$ and such that

$$\left(\frac{(2\alpha+1)\sqrt{C}}{4}\,\frac{(\hat{S}_f(\sigma\|\rho)-\hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}))^{1/2}}{1+\|\Gamma\|_\infty}\right)^{\frac{1}{1+\alpha}}\leq 1.$$

Then, there is a constant $L_\alpha>0$ such that

$$\hat{S}_f(\sigma\|\rho)-\hat{S}_f(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}})\geq$$
$$\geq\frac{L_\alpha}{C}\left(1+\|\Gamma\|_\infty\right)^{-(4\alpha+2)}\|\Gamma\|_\infty^{-(2\alpha+2)}\left\|\sigma^{-1}\right\|_\infty^{-(2\alpha+2)}\left\|\rho-\sigma\sigma_{\mathcal{N}}^{-1}\rho_{\mathcal{N}}\right\|_2^{4(\alpha+1)}.$$


thank you
danke
рахмат
謝謝
ngiyabonga
شكراً جزيلاً
teşekkür ederim
Баярлалаа
спасибо
gracias
dziękuje
obrigado
bedankt
go raibh maith agat
mochchakkeram
sukriya
kop khun krap
grazie
arigatô
dakujem
merci
мерси
terima kasih
감사합니다
ευχαριστώ
ありがとう
tapadh leat
dank je
paldies
grazzi
mahalo
хвала
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