Quasi-factorization of the relative entropy

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QUITEMAD Quantum Information Technologies in Madrid

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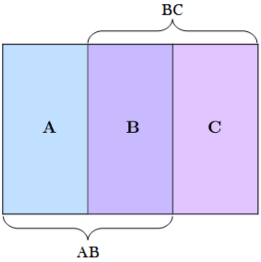
Statement of the problem



Problem

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in S_{ABC}$. Can we prove something like

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right] \ ?$$

Quantum relative entropy

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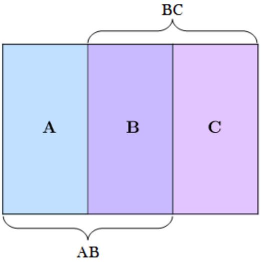

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Classical case, Dai Pra et al. '02

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{1 - 4\|h-1\|_\infty} \, \mu \left[\mathrm{Ent}_\mu(f \mid \mathcal{F}_1) + \mathrm{Ent}_\mu(f \mid \mathcal{F}_2) \right],$$

where $h = \dfrac{d\mu}{d\bar{\mu}}$.

Classical entropy and conditional entropy

Entropy:

$$\mathrm{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Conditional entropy:

$$\mathrm{Ent}_\mu(f \mid \mathcal{G}) = \mu(f \log f \mid \mathcal{G}) - \mu(f \mid \mathcal{G}) \log \mu(f \mid \mathcal{G}).$$

Relative entropy

Quantum relative entropy

Let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$. The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

$$D(\rho_\Lambda || \sigma_\Lambda) = \operatorname{tr}\left[\rho_\Lambda(\log \rho_\Lambda - \log \sigma_\Lambda)\right].$$

Properties of the relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** $\rho_{AB} \mapsto D(\rho_{AB} || \sigma_{AB})$ is continuous.
2. **Additivity.** $D(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B) = D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B)$.
3. **Superadditivity.** $D(\rho_{AB} || \sigma_A \otimes \sigma_B) \geq D(\rho_A || \sigma_A) + D(\rho_B || \sigma_B)$.
4. **Monotonicity.** $D(\rho_{AB} || \sigma_{AB}) \geq D(T(\rho_{AB}) || T(\sigma_{AB}))$ for every quantum channel T.

Characterization of the RE, Wilming et al. '17, Matsumoto '10

If $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$ satisfies $1-4$, then f is the relative entropy.

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Conditional relative entropy

Conditional relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a **conditional relative entropy** in A as a function

$$D_A(\cdot||\cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and
   (2.1) $D_A(\rho_{AB}||\sigma_{AB})=0$ if, and only if, $\rho_{AB} = \sigma_{AB}^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma_{AB}^{1/2}$.
3. **Semi-superadditivity:** $D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$ and
   (3.1) **Semi-additivity:** if $\rho_{AB} = \rho_A \otimes \rho_B$,
   $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.
4. **Semi-motonicity:** For every quantum channel $\mathcal{T}$,

$$D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) + D_B((\operatorname{tr}_A \circ\mathcal{T})(\rho_{AB})||(\operatorname{tr}_A \circ\mathcal{T})(\sigma_{AB}))$$
$$\leq D_A(\rho_{AB}||\sigma_{AB}) + D_B(\operatorname{tr}_A(\rho_{AB})||\operatorname{tr}_A(\sigma_{AB})).$$

Remark

Consider for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$

$$D^+_{A,B}(\rho_{AB}||\sigma_{AB}) = D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}).$$

Then, $D^+_{A,B}$ verifies the following properties:

1. **Continuity:** $\rho_{AB} \mapsto D^+_{A,B}(\rho_{AB}||\sigma_{AB})$ is continuous.
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However, it does not satisfy the property of monotonicity.

Axiomatic characterization of the conditional relative entropy

The only possible conditional relative entropy is given by:

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B)$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

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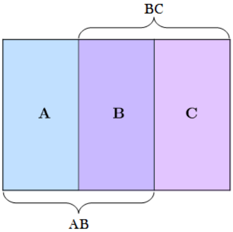

Figure: Choice of indices in $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Result of **quasi-factorization** of the relative entropy, for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$:

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

Quasi-factorization for the CRE

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \frac{1}{1 - 2\|H(\sigma_{AC})\|_\infty} \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \, \sigma_{AC} \, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Note that $H(\sigma_{AC}) = 0$ if σ_{AC} is a tensor product between A and C.

$$
\begin{aligned}
(1 - 2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \leq \\
D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) = \\
= 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).
\end{aligned}
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$$\Leftrightarrow$$

$$(1 + 2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

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This result is equivalent to:

$$\boxed{(1+2\|H(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

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Relation with the classical case

	STATES		OBSERVABLES
QUANTUM SETTING	$D(\rho_{AB} \| \| \sigma_{AB})$ $D(\rho_{AB} \| \| \sigma_{AB}) - D(\rho_B \| \| \sigma_B)$	$f_{AB} = \Gamma^{-1}_{\sigma_{AB}}(\rho_{AB})$ ↔ $f_B = \Gamma^{-1}_{\sigma_B}(\rho_B)$	$\operatorname{tr}[\sigma_{AB} f_{AB} \log f_{AB}]$ $\operatorname{tr}[\operatorname{tr}_A[\sigma_{AB} f_{AB} \log f_{AB}] - \sigma_B f_B \log f_B]$
	↕ $\rho_{AB} \equiv \nu$ $\sigma_{AB} \equiv \mu$		↕ $\operatorname{tr}[\sigma_{AB} \cdot] \equiv \mu(\cdot)$ $\operatorname{tr}_A[\cdot] \equiv \mu(\cdot \mid \mathcal{G})$
CLASSICAL SETTING	$H(\nu, \mu)$ $H_{\mathcal{G}}(\nu, \mu)$	$f = \frac{d\nu}{d\mu}$ ↔	$\mu(f \log f)$ $\mu(\mu(f \log f \mid \mathcal{G}) - \mu(f \mid \mathcal{G}) \log \mu(f \mid \mathcal{G}))$

Figure: Identification between classical and quantum quantities when the states considered are classical.

Application

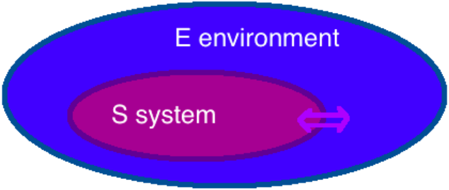

Figure: An open quantum many-body system.

- Interesting for information processing ⇒ Open (unavoidable interactions).
- Dynamics of S is dissipative!
- The continuous-time evolution of a state on S is given by a **quantum Markov semigroup**.

Dissipative quantum systems

Dissipative quantum systems

A **dissipative quantum system** is a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t^*(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$

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Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

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Log-Sobolev constant

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Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ. We define the **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ by

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

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Quantum spin lattices

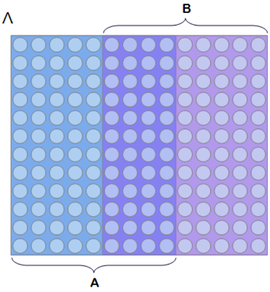

Figure: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

Problem

For a certain $\mathcal{L}_\Lambda^*$, can we prove $\alpha(\mathcal{L}_\Lambda^*) > 0$ using the result of quasi-factorization of the relative entropy?

(1) Quasi-factorization of the relative entropy.

+

(2) Recursive geometric argument.
Lower bound for the log-Sobolev constant in terms of a conditional log-Sobolev constant.

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General quasi-factorization for σ a tensor product

Let $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ and $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. The following inequality holds:

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda). \tag{1}$$

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Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x\in\Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

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For $x \in \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in x by

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where σ_Λ is the fixed point of the evolution, and $D_x(\rho_\Lambda||\sigma_\Lambda)$ is the conditional relative entropy.

Lemma

$$\alpha_\Lambda(\mathcal{L}_x^*) \geq \frac{1}{2}. \tag{3}$$

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where σ_Λ is the fixed point of the evolution, and $D_x(\rho_\Lambda||\sigma_\Lambda)$ is the conditional relative entropy.

Lemma

$$\alpha_\Lambda(\mathcal{L}_x^*) \geq \frac{1}{2}. \qquad (3)$$

$$\begin{aligned} D(\rho_\Lambda || \sigma_\Lambda) &\leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \\ &\leq \sum_{x \in \Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_x^*)} \\ &\leq \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x \in \Lambda} - \operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\ &= \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]\right) \\ &\leq \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]\right). \end{aligned} \tag{2}$$

Positive log-Sobolev constant

$$\alpha(\mathcal{L}_\Lambda^*) \geq \frac{1}{2}.$$

$$\begin{aligned} D(\rho_\Lambda || \sigma_\Lambda) &\leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \\ &\leq \sum_{x \in \Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_x^*)} \\ &\leq \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x \in \Lambda} - \operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\ &= \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \right) \\ &\leq \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \right). \end{aligned} \tag{2}$$

Positive log-Sobolev constant

$$\alpha(\mathcal{L}_\Lambda^*) \geq \frac{1}{2}.$$

Open problems

Problem 1

Can we use any of the quasi-factorization results to prove log-Sobolev constants in a more general setting?

(Kastoryano-Brandao, '15) The heat-bath dynamics, with σ_Λ the Gibbs state of a commuting Hamiltonian, has positive spectral gap. $\Rightarrow$ Log-Sobolev constant?

Problem 2

Is there a better definition for conditional relative entropy?

For further knowledge, Arxiv: 1705.03521 and 1804.09525