# On the modified logarithmic Sobolev inequality for the Heat-Bath dynamics for 1D systems

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**Angelo Lucia** (Caltech),
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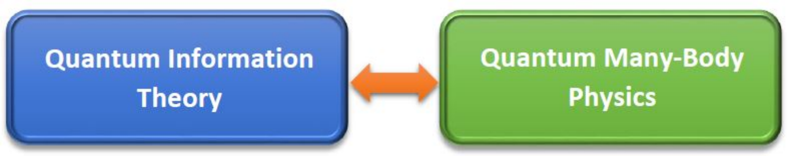
**XV Conference on the Theory of Quantum Computation, Communication and Cryptography, 10 June 2020**

Technical University of Munich

Munich Center for Quantum Science and Technology

# Quantum




Communication channels $\longleftrightarrow$ Physical interactions

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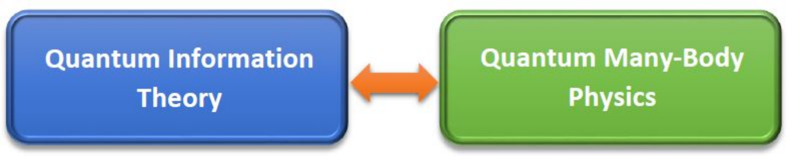


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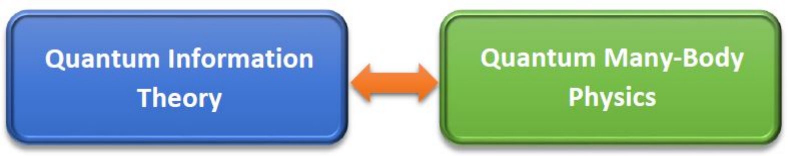


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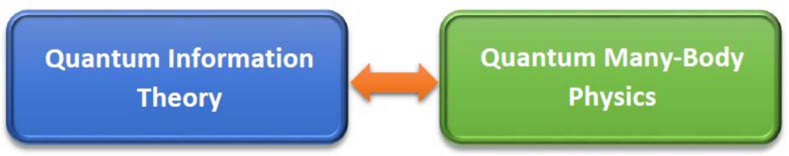


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## Field of study

Dissipative evolutions of quantum many-body systems

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Velocity of convergence of certain quantum dissipative evolutions to their thermal equilibriums.

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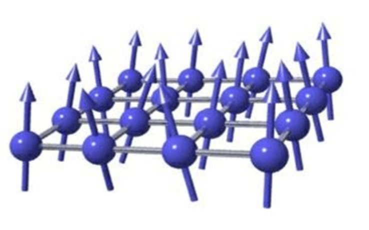
# Contents

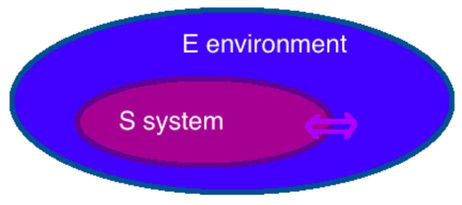
# 1. Quantum dissipative systems



# Open quantum systems

**No experiment can be executed at zero temperature or be completely shielded from noise.**

$\Rightarrow$ Open quantum many-body systems.

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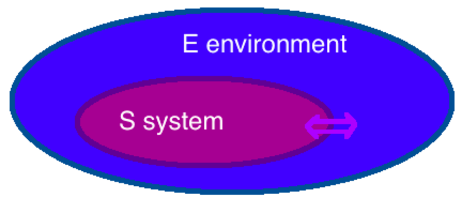


Figure: An open quantum many-body system.

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- The continuous-time evolution of a state on $S$ is given by a q. Markov semigroup (**Markovian approximation**).

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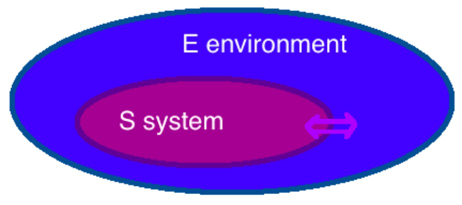


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# Notation

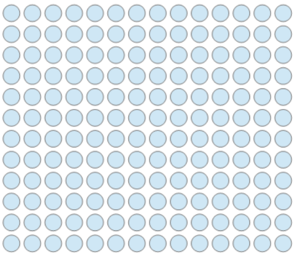

Figure: A quantum spin lattice system.

- Finite lattice $\Lambda \subset\subset \mathbb{Z}^d$.
- To every site $x \in \Lambda$ we associate $\mathcal{H}_x$ $(= \mathbb{C}^D)$.
- The global Hilbert space associated to $\Lambda$ is $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$.
- The set of bounded linear endomorphisms on $\mathcal{H}_\Lambda$ is denoted by $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$.
- The set of density matrices is denoted by $\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{\rho_\Lambda \in \mathcal{B}_\Lambda \ : \ \rho_\Lambda \geq 0 \text{ and } \operatorname{tr}[\rho_\Lambda] = 1\}$.

# Quantum dissipative systems

## Quantum dissipative systems

A **quantum dissipative system** is a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

**Semigroup:**

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The infinitesimal generator $\mathcal{L}_\Lambda^*$ of the previous semigroup of quantum channels is usually called **Lindbladian**.

$$\mathcal{T}_t^* = e^{t\mathcal{L}_\Lambda^*} \Leftrightarrow \mathcal{L}_\Lambda^* = \frac{d}{dt}\mathcal{T}_t^* \mid_{t=0} .$$

**Notation:** $\rho_t := \mathcal{T}_t^*(\rho)$.

$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t^*(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$

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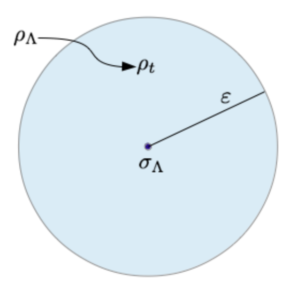
## Mixing time

We define the **mixing time** of $\{\mathcal{T}_t^*\}$ by

$$\tau(\varepsilon) = \min\left\{ t > 0 : \sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \|\mathcal{T}_t^*(\rho) - \mathcal{T}_\infty^*(\rho)\|_1 \leq \varepsilon \right\}.$$



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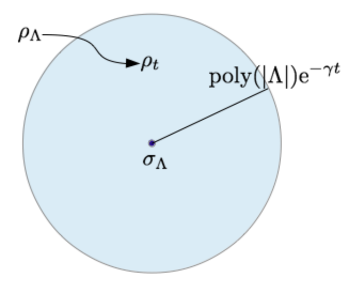


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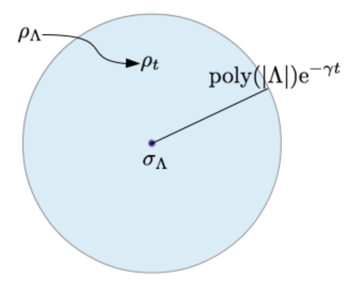


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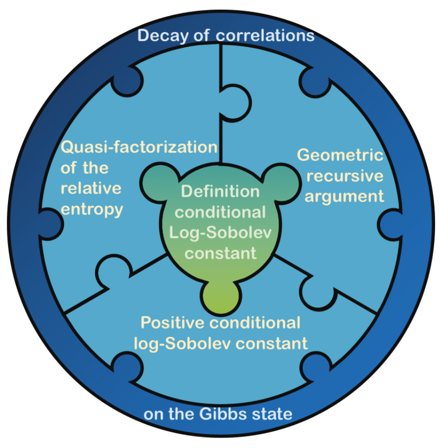
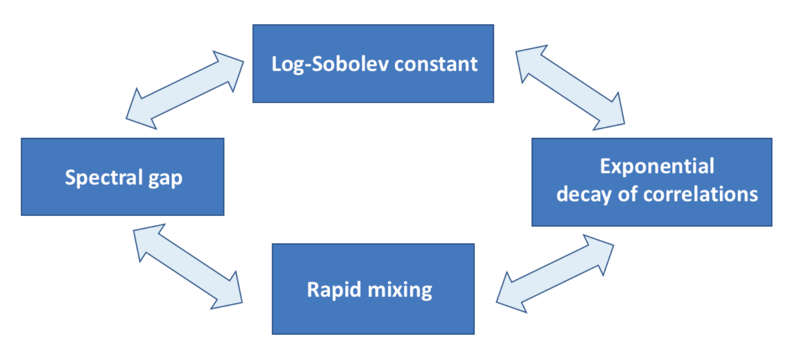
# 2. General strategy for log-Sobolev inequalities



# Classical spin systems



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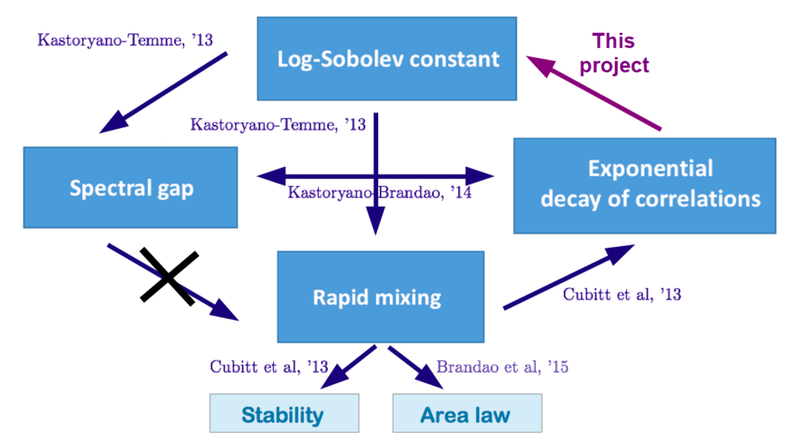

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**Main problem of this talk**

Develop a strategy to find positive log-Sobolev constants.

**Concrete problem**

Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant for the heat-bath dynamics in 1D.

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(Cesi, Dai Pra-Paganoni-Posta, '02)

(1) Quasi-factorization of the entropy (in terms of a conditional entropy).

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(2) Recursive geometric argument.
Lower bound for the global log-Sobolev constant in terms of the log-Sobolev constant of a size-fixed region.

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# Conditional log-Sobolev constant

## Log-Sobolev constant

Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state $\sigma_\Lambda$. We define the **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ by

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

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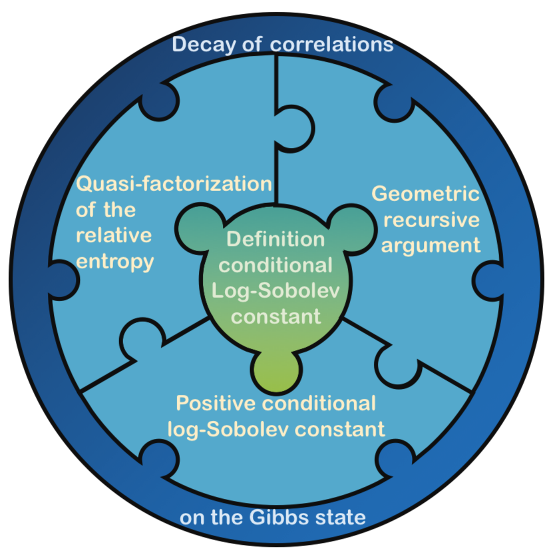
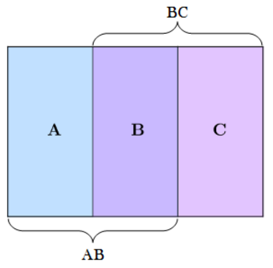
# Strategy


Decay of correlations
Quasi-factorization of the relative entropy
Geometric recursive argument
Definition conditional Log-Sobolev constant
Positive conditional log-Sobolev constant
on the Gibbs state

# Quasi-factorization of the relative entropy

The strategy is based on a solution for the following problem.



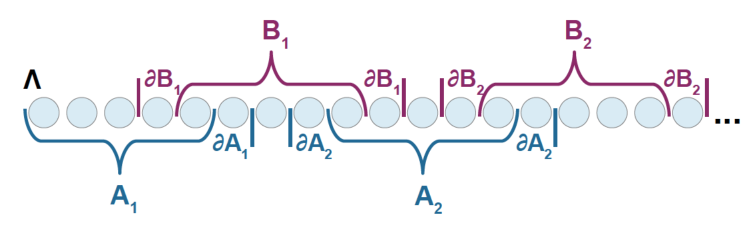

**Problem**

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in S_{ABC}$. Can we prove something like

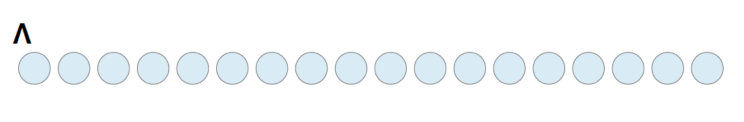
$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[ D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right]$$

where $\xi(\sigma_{ABC})$ depends only on $\sigma_{ABC}$ and measures how far $\sigma_{AC}$ is from $\sigma_A \otimes \sigma_C$?

# 3. Log-Sobolev inequality for the heat-bath dynamics for 1D system



# Log-Sobolev inequality for the heat-bath dynamics




**The dynamics:** For every $\rho_\Lambda \in \mathcal{S}_\Lambda$,

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) := \sum_{x \in \Lambda} \left( \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} - \rho_\Lambda \right).$$

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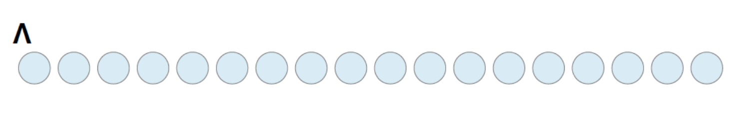


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$$\mathcal{L}_\Lambda^*(\rho_\Lambda) := \sum_{x\in\Lambda} \left( \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} - \rho_\Lambda \right).$$

Given $A \subset \Lambda$, can we prove something like

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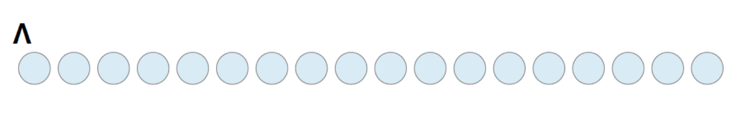
# Log-Sobolev inequality for the heat-bath dynamics

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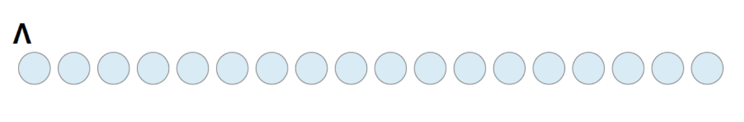


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# Conditional relative entropy

## Classical entropy and conditional entropy

Entropy:

$$\mathrm{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Conditional entropy:

$$\mathrm{Ent}_\mu(f \mid \mathcal{G}) = \mu(f(\log f - \log \mu(f \mid \mathcal{G})) \mid \mathcal{G}).$$

## Quantum relative entropy

The **quantum relative entropy** of $\rho_\Lambda$ and $\sigma_\Lambda$ is defined by:

$$D(\rho_\Lambda || \sigma_\Lambda) = \mathrm{tr}\left[\rho_\Lambda (\log \rho_\Lambda - \log \sigma_\Lambda)\right].$$

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for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

C.-Lucia-Pérez García, '18 → Axiomatic characterization of the CRE.

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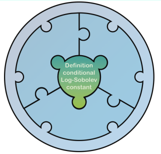
# Conditional log-Sobolev constant




## Conditional log-Sobolev constant

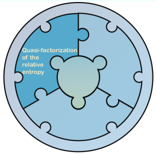
For $A \subset \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in $A$ by

$$\alpha_\Lambda(\mathcal{L}_A^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_A^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2 D_A(\rho_\Lambda || \sigma_\Lambda)},$$

where $\sigma_\Lambda$ is the fixed point of the evolution, and

$$D_A(\rho_\Lambda || \sigma_\Lambda) = D(\rho_\Lambda || \sigma_\Lambda) - D(\rho_{A^c} || \sigma_{A^c}).$$

# Quasi-factorization of the relative entropy



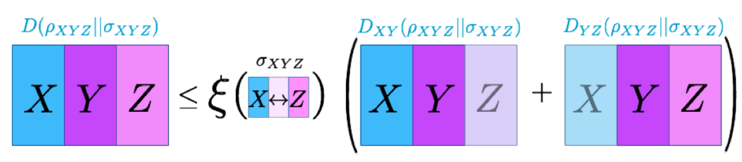

## Quasi-factorization for the CRE, C.-Lucia-Pérez García '18

Let $\mathcal{H}_{XYZ}$ and $\rho_{XYZ}, \sigma_{XYZ} \in \mathcal{S}_{XYZ}$. The following holds

$$D(\rho_{XYZ}||\sigma_{XYZ}) \leq \xi(\sigma_{XZ}) \left[ D_{XY}(\rho_{XYZ}||\sigma_{XYZ}) + D_{YZ}(\rho_{XYZ}||\sigma_{XYZ}) \right],$$

where

$$\xi(\sigma_{XZ}) = \frac{1}{1 - 2\left\| \sigma_X^{-1/2} \otimes \sigma_Z^{-1/2} \, \sigma_{XZ} \, \sigma_X^{-1/2} \otimes \sigma_Z^{-1/2} - \mathbb{1}_{XZ} \right\|_\infty}.$$



This result is equivalent to:

$$\boxed{(1+2\|H(\sigma_{XY})\|_\infty)D(\rho_{XY}||\sigma_{XY}) \geq D(\rho_X||\sigma_X) + D(\rho_Y||\sigma_Y)}.$$

Recall:

- **Superadditivity.** $D(\rho_{XY}||\sigma_X \otimes \sigma_Y) \geq D(\rho_X||\sigma_X) + D(\rho_Y||\sigma_Y)$.

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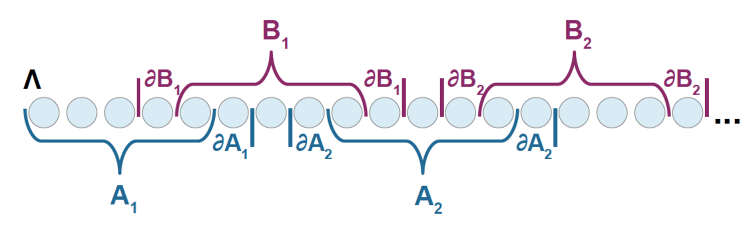
# Quasi-factorization of the relative entropy

**STEP 1**




$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{j=1}^{n} B_j$$

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2\|H(\sigma_{A^c B^c})\|_\infty} \left[ D_A(\rho_\Lambda || \sigma_\Lambda) + D_B(\rho_\Lambda || \sigma_\Lambda) \right],$$

$$H(\sigma_{A^c B^c}) := \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^c B^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - \mathbb{1}_{A^c B^c}.$$

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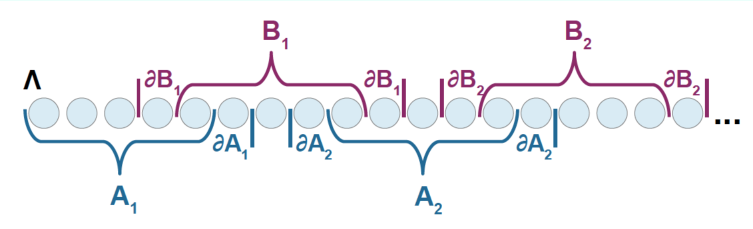


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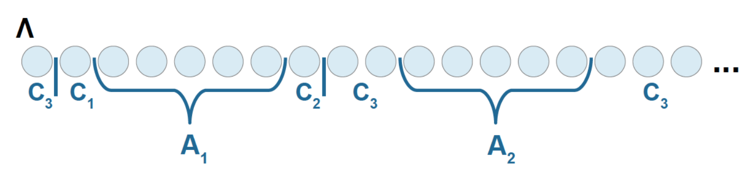
# Sketch of the proof

**STEP 2**




$$D_A(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{i=1}^{n} D_{A_i}(\rho_\Lambda || \sigma_\Lambda)$$

$\sigma_\Lambda$ is a QMC between $A_1 \leftrightarrow \partial A_1 \leftrightarrow \Lambda \setminus (A_1 \cup \partial A_1)$

$$\sigma_\Lambda = \bigoplus_{i \in I} \sigma_{A_1 (\partial a_1)_i^L} \otimes \sigma_{(\partial a_1)_i^R \Lambda \setminus (A_1 \cup \partial A_1)}$$

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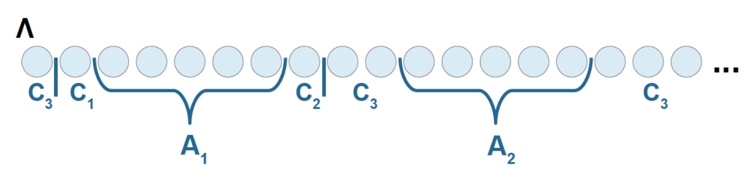


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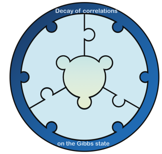
# Clustering of correlations on the Gibbs state




## Assumption 1

In a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$, $A$ and $B$ not connected, we have

$$\|h(\sigma_{AB})\|_\infty = \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty \leq K < \frac{1}{2}.$$

In particular, Gibbs states at high enough temperature satisfy this.

## Assumption 2

For any $B \subset \Lambda$, $B = B_1 \cup B_2$, it holds:

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In particular, tensor products satisfy this (with $f = 1$).

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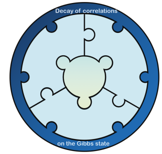


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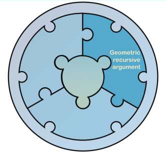
# Geometric recursive argument




**STEP 3**

Using locality of the Lindbladian

$$\mathcal{L}_A^* + \mathcal{L}_B^* = \mathcal{L}_{A\cup B}^* + \mathcal{L}_{A\cap B}^*$$

and quasi-factorization:

$$\text{Assumption 1} \Rightarrow \alpha(\mathcal{L}_\Lambda^*) \geq \tilde{K} \min_{i\in\{1,\ldots n\}} \left\{ \alpha_\Lambda(\mathcal{L}_{A_i}^*), \alpha_\Lambda(\mathcal{L}_{B_i}^*) \right\}$$

Recursion appears in a possible extension to larger dimension.

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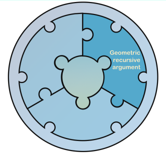


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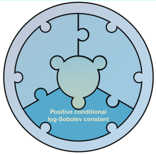
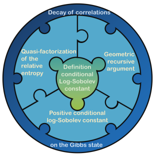



**STEP 4**

$$\text{Assumption 2} \Rightarrow \alpha_\Lambda(\mathcal{L}^*_{A_i}) \geq g(\sigma_{A_i\partial}) > 0.$$

# Heat-bath dynamics in 1D




**Theorem, Bardet-C.-Lucia-Pérez García-Rouzé '19**

In 1D, if Assumptions 1 and 2 hold, for a $k$-local commuting Hamiltonian, the heat-bath dynamics has a positive log-Sobolev constant.

**Previous results:**

- **Kastoryano-Brandao '15.** In 1D, for a $k$-local commuting Hamiltonian, the heat-bath dynamics is always gapped.

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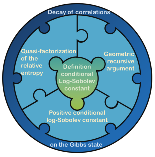


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# Open problems

**Problem 1**

Does this hold for larger dimension?

**Problem 2**

Is there a better definition for conditional relative entropy?

**Problem 3**

Can we do something similar for different dynamics?


thank you
danke
рахмат
謝謝
ngiyabonga
شكراً جزيلاً
teşekkür ederim
gracias
Баярлалаа
спасибо
faafetai lava
mersi
barka
welalin
tack
spas
dank je
misaotra
matondo
paldies
grazzi
mahalo
tapadh leat
vinaka
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blagodaram
kiitos
dankie
dhanyavad
nanni
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bayarlalaa
gràcie
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mauruuru
köszönöm
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