Quasi-factorization of the quantum relative entropy

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- $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ (or $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$).
- $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$, set of bounded linear operators.
- $\mathcal{A}_\Lambda \subseteq \mathcal{B}_\Lambda$, set of Hermitian operators.
- $\mathcal{S}_\Lambda := \{ f \in \mathcal{A}_\Lambda \, : \, f \geq 0 \text{ and } \operatorname{tr}[f] = 1 \}$.
- $f \in \mathcal{B}_\Lambda$ has support on $A \subseteq \Lambda$ if $f = f_A \otimes \mathbb{1}_B$ for certain $f_A \in \mathcal{B}_A$.
- Modified partial trace: $\operatorname{tr}_A : f \mapsto \operatorname{tr}_A[f] \otimes \mathbb{1}_A$, where $\operatorname{tr}_A[f]$ has support in B.
- We denote by f_B the observable $\operatorname{tr}_A[f]$ with support in B.

Quantum relative entropy

Let $f, g \in \mathcal{A}_\Lambda$, f verifying $\operatorname{tr}[f] \neq 0$. The **quantum relative entropy** of f and g is defined by:

$$D(f||g) = \frac{1}{\operatorname{tr}[f]} \operatorname{tr}\left[f(\log f - \log g)\right]. \quad (1)$$

Remark

In this talk, we only consider density matrices (with trace 1). In this case, the **quantum relative entropy** is given by:

$$D(\rho||\sigma) = \operatorname{tr}\left[\rho(\log \rho - \log \sigma)\right]. \quad (2)$$

Properties of the relative entropy

Let $\mathcal{H}_{AB}$ be a bipartite finite dimensional Hilbert space, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Non-negativity.** $D(\rho_{AB}||\sigma_{AB}) \geq 0$ and $D(\rho_{AB}||\sigma_{AB}) = 0 \Leftrightarrow \rho_{AB} = \sigma_{AB}$.
2. **Finiteness.** $D(\rho_{AB}||\sigma_{AB}) < \infty$ if, and only if, $\text{supp}(\rho_{AB}) \subseteq \text{supp}(\sigma_{AB})$, where supp stands for support.
3. **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.
4. **Factorization.** $D(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
5. **Joint convexity.** $D(\rho_{AB}||\sigma_{AB}) \leq p_1 D(\rho^1_{AB}||\sigma^1_{AB}) + p_2 D(\rho^2_{AB}||\sigma^2_{AB})$ if $\rho_{AB} = p_1\, \rho^1_{AB} + p_2\, \rho^2_{AB}$ and $\sigma_{AB} = p_1\, \sigma^1_{AB} + p_2\, \sigma^2_{AB}$.

Problem

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in S_{AB}$. Can we prove something like

$$D(\rho_{AB}||\sigma_{AB}) \leq C\left[D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB})\right] \text{ ?}$$

Yes! (We will see how later)

Classical entropy and conditional entropy

Consider a probability space $(\Omega, \mathcal{F}, \mu)$ and define, for every $f > 0$, the **entropy** of f by

$$\mathrm{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Given a σ-field $\mathcal{G} \subseteq \mathcal{F}$, we define the **conditional entropy** of f in $\mathcal{G}$ by

$$\mathrm{Ent}_\mu(f \mid \mathcal{G}) = \mu(f \log f \mid \mathcal{G}) - \mu(f \mid \mathcal{G}) \log \mu(f \mid \mathcal{G}).$$

With these definitions, the following lemma is proven:

Lemma

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and $\mathcal{F}_1, \mathcal{F}_2$ sub-σ-fields of $\mathcal{F}$. Suppose that there exists a probability measure $\bar{\mu}$ that makes $\mathcal{F}_1$ and $\mathcal{F}_2$ independent, $\mu \ll \bar{\mu}$ and $\mu \mid \mathcal{F}_i = \bar{\mu} \mid \mathcal{F}_i$ for $i = 1, 2$. Then, for every $f \geq 0$ such that $f \log f \in L^1(\mu)$ and $\mu(f) = 1$,

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{1 - 4\|h - 1\|_\infty} \, \mu \left[\mathrm{Ent}_\mu(f \mid \mathcal{F}_1) + \mathrm{Ent}_\mu(f \mid \mathcal{F}_2) \right],$$

where $h = \dfrac{d\mu}{d\bar{\mu}}$.

Conditional relative entropy by differences

$$D_A^D(\rho||\sigma) =$$
$$\operatorname{tr}[\operatorname{tr}_A[\rho(\log\rho - \log\sigma)] - \operatorname{tr}_A[\rho](\log\operatorname{tr}_A[\rho] - \log\operatorname{tr}_A[\sigma])],$$

Conditional relative entropy by differences

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and let $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. We define the **conditional relative entropy by differences** of ρ_{AB} and σ_{AB} in A by:

$$D_A^D(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B).$$

Properties

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. The following properties hold:

1. $D_A^D(\rho_{AB}||\sigma_{AB}) \geq 0$ for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.
2. If $\rho_{AB} = \sigma_{AB}$, then $D_A^D(\rho_{AB}||\sigma_{AB}) = 0$.

Conditional expectation

Conditional expectation

Let $\mathcal{A}$ and $\mathcal{B}$ be two matrix algebras, and σ a full rank state on $\mathcal{A} \otimes \mathcal{B}$. A map $\mathbb{E} : \mathcal{A} \otimes \mathcal{B} \to \mathcal{B}$ will be called a **conditional expectation of σ on $\mathcal{B}$** if it satisfies the following:

1. **Complete positivity.** $\mathbb{E}$ is completely positive and unital.
2. **Consistency.** For every $f \in \mathcal{A} \otimes \mathcal{B}$, $\operatorname{tr}[\sigma \mathbb{E}(f)] = \operatorname{tr}[\sigma f]$.
3. **Reversibility.** For every $f, g \in \mathcal{A} \otimes \mathcal{B}$, $\langle \mathbb{E}(f), g \rangle_\sigma = \langle f, \mathbb{E}(g) \rangle_\sigma$.
4. **Monotonicity.** For every $f \in \mathcal{A} \otimes \mathcal{B}$ and $n \in \mathbb{N}$, $\langle \mathbb{E}^n(f), f \rangle_\sigma \geq \langle \mathbb{E}^{n+1}(f), f \rangle_\sigma$.

Remark

1. $\mathbb{E}^*(\sigma) = \sigma$, where the dual is taken with respect to the Hilber-Schimdt scalar product.
2. $\mathbb{E}$ is self-adjoint in $L_2(\sigma)$.

Minimal conditional expectation

We define the **minimal conditional expectation** of σ on A by

$$\mathbb{E}_A^\sigma(\rho_{AB}) := \operatorname{tr}_A[\eta_A^\sigma \, \rho_{AB} \, \eta_A^{\sigma\dagger}], \tag{3}$$

where $\eta_A^\sigma := (\operatorname{tr}_A[\sigma_{AB}])^{-1/2}\sigma_{AB}^{1/2}$.

$(\mathbb{E}_A^\sigma)^*$ (hereafter denoted by $\mathbb{E}_A^*$) is given by

$$\mathbb{E}_A^*(\rho_{AB}) := \sigma_{AB}^{1/2} \, \sigma_B^{-1/2} \, \rho_B \, \sigma_B^{-1/2} \, \sigma_{AB}^{1/2}. \tag{4}$$

Conditional relative entropy by expectations

Conditional relative entropy by expectations

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a composite Hilbert space and $\rho_{AB}, \sigma_{AB} \in S_{AB}$. Let $\mathbb{E}$ be a conditional expectation. We define the **conditional relative entropy by expectations** of ρ_{AB} and σ_{AB} in A by:

$$D_A^E(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\mathbb{E}_A^*(\rho_{AB})).$$

Properties

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a composite Hilbert space. The following properties hold:

1. $D_A^E(\rho_{AB}||\sigma_{AB}) \geq 0$ for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.
2. If $\rho_{AB} = \sigma_{AB}$, then $D_A^E(\rho_{AB}||\sigma_{AB}) = 0$.

Problem

Under which conditions holds

$$D_A^D(\rho||\sigma) = D_A^E(\rho||\sigma)?$$

Example

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. If $\sigma = \sigma_A \otimes \sigma_B$, then

$$D_A^D(\rho||\sigma) = D_A^E(\rho||\sigma)$$

for every $\rho \in \mathcal{S}_\Lambda$, $A \subseteq \Lambda$.

In general, it is an open question.

Quasi-factorization results

Conditional relative entropy by differences

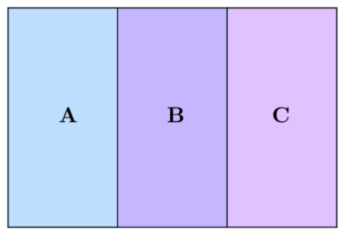

Figura: The set of indices of a tripartite Hilbert space $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Quasi-factorization

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ be a tripartite Hilbert space and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$(1 - 2\|h\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \leq$$
$$\leq D^D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D^D_{BC}(\rho_{ABC}||\sigma_{ABC}),$$

where

$$h = \frac{1}{2}\{\sigma_A^{-1} \otimes \sigma_C^{-1}, \sigma_{AC}\} - \mathbb{1}_{AC}.$$

Note that $h = 0$ if σ is a tensor product between A and C.

Quasi-factorization for conditional expectations

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ be a tripartite Hilbert space and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$(1 - 2\|h\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \leq$$
$$\leq D^E_{AB}(\rho_{ABC}||\sigma_{ABC}) + D^E_{BC}(\rho_{ABC}||\sigma_{ABC}),$$

where

$$h = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \, \sigma_{AC} \, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC},$$

and $\mathbb{E}$ is the minimal conditional expectation. Note that $h = 0$ if σ is a tensor product between A and C.

Step 1

For density matrices $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$, it holds that

$$D(\rho_{ABC}||\sigma_{ABC}) \leq$$
$$\leq D^E_{AB}(\rho_{ABC}||\sigma_{ABC}) + D^E_{BC}(\rho_{ABC}||\sigma_{ABC}) + \log \operatorname{tr} M,$$

where $M = \exp\left[-\log \sigma_{ABC} + \log \mathbb{E}^*_{AB}(\rho_{ABC}) + \log \mathbb{E}^*_{BC}(\rho_{ABC})\right]$ and equality holds (both sides being equal to zero) if $\rho_{ABC} = \sigma_{ABC}$.
Moreover, if B is an empty set and $\sigma_{AC} = \sigma_A \otimes \sigma_C$, then $\log \operatorname{tr} M = 0$.

Step 2

With the same notation of step 1, we have that

$$\log \operatorname{tr} M \leq \operatorname{tr}(h\, \rho_A \otimes \rho_C), \tag{5}$$

where

$$h = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \sigma_{AC}\, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Step 3

With the same notation of the previous steps,

$$\operatorname{tr}[h\, \rho_A \otimes \rho_C] \leq 2\|h\|_\infty D(\rho_{ABC}||\sigma_{ABC}). \tag{6}$$

Motivation

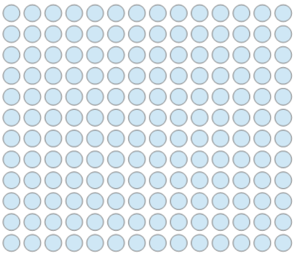

Figura: A quantum spin lattice system.

- Lattice $\Lambda \subseteq \mathbb{Z}^d$.
- For every site x, $\mathcal{H}_x$ $(= \mathbb{C}^d)$.
- The global Hilbert space associated to Λ is $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$.

Rapid mixing

Dissipative quantum systems

A **dissipative quantum system** is a 1-parameter continuous semigroup $\{\mathcal{T}_t\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{B}_\Lambda$.

- **Positive:** Maps positive operators to positive operators.
- **Completely positive:** $\mathcal{T} \otimes \mathbb{1} : \mathcal{B}_\Lambda \otimes \mathcal{M}_n \to \mathcal{B}_\Lambda \otimes \mathcal{M}_n$ is positive $\forall n \in \mathbb{N}$.
- **Trace preserving:** $\operatorname{tr}[\mathcal{T}(f)] = \operatorname{tr}[f] \; \forall f \in \mathcal{B}_\Lambda$.

LIOUVILLIAN

The infinitesimal generator $\mathcal{L}$ of a semigroup of quantum channels is called **Liouvillian**.

$$\mathcal{T}_t = e^{t\mathcal{L}} \Leftrightarrow \mathcal{L} = \frac{d}{dt}\mathcal{T}_t \mid_{t=0}$$

CONTRACTION

We define the **contraction** of $\mathcal{T}_t$ by

$$\eta(\mathcal{T}_t) = \frac{1}{2} \sup_{\rho \in \mathcal{S}_\Lambda} \left\| \mathcal{T}_t(\rho) - \mathcal{T}_\infty(\rho) \right\|_1 .$$

RAPID MIXING

We say that $\mathcal{L}$ satisfies **rapid mixing** if

$$\eta(\mathcal{T}_t) \leq \text{poly}(|\Lambda|) e^{-\gamma t} .$$

Log-Sobolev inequality

Let σ be the stationary state of a semigroup generated by the quantum dynamical master equation

$$\partial_t \rho_t = \mathcal{L}^*(\rho_t), \tag{7}$$

where $\mathcal{L}$ is the Liouvillian in the Heisenberg picture.

We define the relative entropy of ρ_t and σ by:

$$D(\rho_t||\sigma) = \operatorname{tr}[\rho_t(\log \rho_t - \log \sigma)]. \tag{8}$$

Therefore, since ρ_t evolves according to $\mathcal{L}^*$, the derivate of $D(\rho_t||\sigma)$ is given by

$$\partial_t D(\rho_t||\sigma) = \operatorname{tr}[\mathcal{L}^*(\rho_t)(\log \rho_t - \log \sigma)], \tag{9}$$

and we want to find a lower bound for the derivative of $D(\rho_t||\sigma)$ in terms of itself:

$$2\alpha D(\rho_t||\sigma) \leq -\operatorname{tr}[\mathcal{L}^*(\rho_t)(\log \rho_t - \log \sigma)]. \tag{10}$$

Log-Sobolev inequality

Let $\mathcal{L} : \mathcal{B}_\Lambda \to \mathcal{B}_\Lambda$ be a primitive reversible Liouvillian with stationary state σ. We define the log-Sobolev constant of $\mathcal{L}$ by

$$S_\Lambda(\mathcal{L}) := \inf_{\rho \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}^*(\rho)(\log \rho - \log \sigma)]}{2D(\rho||\sigma)}$$

Result

If $S_\Lambda(\mathcal{L}) > 0$,

$$\|\rho_t - \sigma\|_1 \leq \sqrt{2\log(1/\sigma_{\text{min}})}e^{-S_\Lambda(\mathcal{L})t}.$$

Log-Sobolev inequality $\Rightarrow$ Rapid mixing.

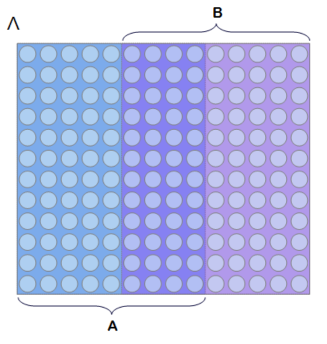

Figura: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

QUASI-FACTORIZATION FOR LATTICES

Let Λ be a finite subset of $\mathbb{Z}^d$ and let $A, B \subseteq \Lambda$ so that $A \cup B = \Lambda$ but they are not necessarily disjoint. Let $\rho, \sigma \in \mathcal{S}_\Lambda$. Then, the following inequality holds

$$(1 - 2\|h_X\|_\infty) D(\rho||\sigma) \leq D_A^X(\rho||\sigma) + D_B^X(\rho||\sigma), \tag{11}$$

where $D_A^X(\rho||\sigma) = D_A^D(\rho||\sigma)$ or $D_A^E(\rho||\sigma)$ and the same for B, and

- For $D_A^X(\rho||\sigma) = D_A^D(\rho||\sigma)$,
$$h_X = \frac{1}{2}\left\{\sigma_{A^c}^{-1} \otimes \sigma_{B^c}^{-1}, \sigma_{A^c \cup B^c}\right\} - \mathbb{1}_{A^c \cup B^c}.$$
- For $D_A^X(\rho||\sigma) = D_A^E(\rho||\sigma)$,
$$h_X = \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^c \cup B^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - \mathbb{1}_{A^c \cup B^c}$$

Note that $h = 0$ if $A \cap B = \emptyset$ and σ is a product.

Log-Sobolev inequality

Since

$$(1-2\|h\|_\infty)D(\rho||\sigma) \leq D_A^D(\rho||\sigma) + D_B^D(\rho||\sigma),$$

defining a conditional log-Sobolev constant in A and B, $S_\Lambda(\mathcal{L}_A)$ and $S_\Lambda(\mathcal{L}_B)$, we have

$$S_\Lambda(\mathcal{L}) \geq C \min_{1\leq i\leq n} \{S_\Lambda(A_i), S_\Lambda(B_i)\}.$$

For further knowledge,
soon on Arxiv.
(we hope so!)