A strengthened data processing inequality for the Belavkin-Staszewski relative entropy

Ángela Capel
Technische Universität München

Joint work with Andreas Bluhm (U. Copenhagen)

Based on arXiv: 1904.10768
Reviews in Mathematical Physics 32 (2), 2050005, 2020

Beyond IID 8
13th November 2020

Main concepts

Relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **relative entropy** is defined as:

$$D(\sigma||\rho) := \operatorname{tr}[\sigma(\log \sigma - \log \rho)].$$

Belavkin-Staszewski relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **BS-entropy** is defined as:

$$D_{\mathrm{BS}}(\sigma||\rho) := \operatorname{tr}\left[\sigma \log \left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right].$$

Relation between relative entropies

The following holds for every $\sigma > 0, \rho > 0$:

$$D_{\mathrm{BS}}(\sigma||\rho) \geq D(\sigma||\rho).$$

Data processing inequality

Quantum channel: $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ CPTP map.

Data processing inequality

$$D(\sigma||\rho) \geq D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)).$$

Conditions for equality, Petz '86

$D(\sigma||\rho) = D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \Leftrightarrow \sigma = \mathcal{P}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma)$, for $\mathcal{P}_{\mathcal{T}}^{\rho}$ a recovery map.

Petz recovery map: $\mathcal{R}_{\mathcal{T}}^{\rho}(\cdot) := \rho^{1/2}\mathcal{T}^* \left(\mathcal{T}(\rho)^{-1/2}(\cdot)\mathcal{T}(\rho)^{-1/2} \right) \rho^{1/2}$.

Strengthened bounds for DPI of RE

Operational meaning of $D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho))$

- **Thermodynamics:** Cost of a certain quantum process (Faist et al, '18).
- **Partial trace:** Conditional relative entropy (C.-Lucia-Pérez García, '18).

DPI for relative entropy: $D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \geq 0.$

Problem

Can we find a lower bound for the DPI in terms of $\mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma)$*?*

Answer: It is not possible **(Brandao et al. '15, Fawzi^2 '17)**.

(Carlen-Vershynina '17) $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ conditional expectation, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$:

$$D(\sigma\|\rho) - D(\sigma_{\mathcal{N}}\|\rho_{\mathcal{N}}) \geq \left(\frac{\pi}{8}\right)^4 \|L_\rho R_{\sigma^{-1}}\|_\infty^{-2} \|\mathcal{R}^{\sigma}_{\mathcal{E}}(\rho_{\mathcal{N}}) - \rho\|_1^4.$$

(Carlen-Vershynina '18) Extension to standard f-divergences.

Questions

BS recovery condition

Can we prove an equivalent condition for equality in DPI for the BS entropy which provides an explicit expression of recovery for σ?

Strengthened DPI for BS entropy

Can we provide a lower bound for the DPI for the BS entropy in terms of a (hypothetical) BS recovery condition?

Equivalent conditions for equality on DPI

$\rho_{\mathcal{T}} := \mathcal{T}(\rho)$, $\sigma_{\mathcal{T}} := \mathcal{T}(\sigma)$

Equivalent conditions for equality on DPI (Bluhm-C. '20)

Let $\mathcal{M}$ and $\mathcal{N}$ be matrix algebras, $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ a quantum channel, $\sigma > 0$, $\rho > 0$ two quantum states on $\mathcal{M}$. The following are equivalent:

1. $D_{\mathrm{BS}}(\sigma \| \rho) = D_{\mathrm{BS}}(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}})$.
2. $\rho = \sigma \mathcal{T}^*(\mathcal{T}(\sigma)^{-1} \mathcal{T}(\rho))$.

BS recovery condition

$$\mathcal{B}^{\sigma}_{\mathcal{T}}(\cdot) := \sigma \mathcal{T}^*(\mathcal{T}(\sigma)^{-1}(\cdot)).$$

Strengthened DPI for the BS-entropy (Bluhm-C. '20)

Let $\mathcal{M}$ and $\mathcal{N}$ be matrix algebras, $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ a quantum channel, $\sigma > 0$, $\rho > 0$ two quantum states on $\mathcal{M}$. Then,

$$D_{\mathrm{BS}}(\sigma \| \rho) - D_{\mathrm{BS}}(\sigma_{\mathcal{T}} \| \rho_{\mathcal{T}}) \geq \left(\frac{\pi}{8} \right)^4 \| \Gamma \|_\infty^{-4} \left\| \sigma^{-1} \right\|_\infty^{-2} \left\| \rho - \sigma \mathcal{T}^* \left(\sigma_{\mathcal{T}}^{-1} \rho_{\mathcal{T}} \right) \right\|_2^4 .$$

Comparison results for the relative entropy and the BS-entropy

Relative entropy	**BS-entropy**
$\operatorname{tr}[\sigma(\log\sigma - \log\rho)]$	$\operatorname{tr}\left[\sigma\log\left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right]$
$\rho = \rho^{1/2}\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1/2}\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1/2}\right)\rho^{1/2}$	$\sigma = \rho\,\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1}\mathcal{T}(\sigma)\right)$
$\left(\frac{\pi}{8}\right)^4 \lVert L_\rho R_{\sigma^{-1}}\rVert_\infty^{-2}\lVert\mathcal{R}_{\mathcal{E}}^{\sigma}(\rho_{\mathcal{N}}) - \rho\rVert_1^4$	$\left(\frac{\pi}{8}\right)^4 \lVert\Gamma\rVert_\infty^{-4}\lVert\sigma^{-1}\rVert_\infty^{-2}\lVert\rho - \mathcal{B}_{\mathcal{T}}^{\sigma}\circ\mathcal{T}(\rho)\rVert_2^4$
Extension to standard f-divergences	Extension to maximal f-divergences

THANK YOU FOR YOUR ATTENTION!