Entropy decay for 1D quantum Gibbs samplers

Ángela Capel (ICMAT-UAM, Madrid)

Joint work with Ivan Bardet (INRIA, Paris), Nilanjana Datta (U. Cambridge), Angelo Lucia (Caltech), Cambyse Rouzé (T. U. München) and David Pérez-García (U. Complutense de Madrid).

Caltech, IQI Seminar, 16th April 2019

BASED ON:

1. A. Capel, A. Lucia and D. Pérez-García, **Superadditivity of Quantum Relative Entropy for General States**, *IEEE Trans. on Inf. Theory*, 64 (7) (2018), 4758–4765.
2. A. Capel, A. Lucia and D. Pérez-García, **Quantum Conditional Relative Entropy and Quasi-Factorization of the Relative Entropy**, *J. Phys. A: Math. Theor.*, 51 (2018), 484001.
3. I. Bardet, A. Capel, N. Datta, A. Lucia, D. Pérez-García and C. Rouzé, **Entropy decay for 1D quantum Gibbs samplers**, in preparation.

Quantum

Q. information theory ⟷ Q. many-body physics

Communication channels ⟷ Physical interactions

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Dissipative evolutions of quantum many-body systems

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Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant.

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Contents

1. Quantum dissipative systems

Open quantum systems

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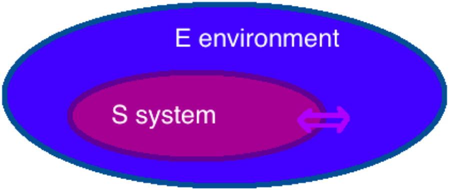
⇒ Open quantum many-body systems.



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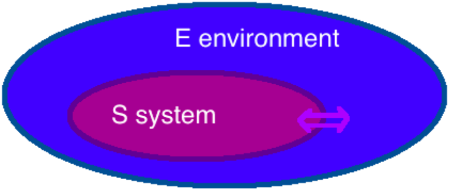

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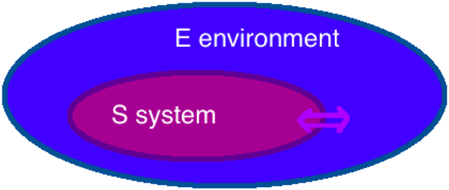

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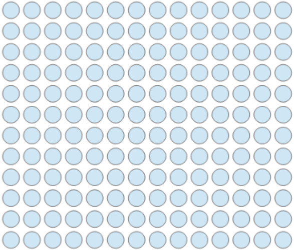

Figure: A quantum spin lattice system.

- Finite lattice $\Lambda \subset\subset \mathbb{Z}^d$.
- To every site $x \in \Lambda$ we associate $\mathcal{H}_x$ $(= \mathbb{C}^D)$.
- The global Hilbert space associated to Λ is $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$.
- The set of bounded linear endomorphisms on $\mathcal{H}_\Lambda$ is denoted by $\mathcal{B}_\Lambda := \mathcal{B}(\mathcal{H}_\Lambda)$.
- The set of density matrices is denoted by $\mathcal{S}_\Lambda := \mathcal{S}(\mathcal{H}_\Lambda) = \{\rho_\Lambda \in \mathcal{B}_\Lambda \, : \, \rho_\Lambda \geq 0 \text{ and } \operatorname{tr}[\rho_\Lambda] = 1\}$.

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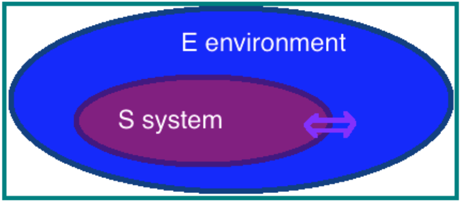

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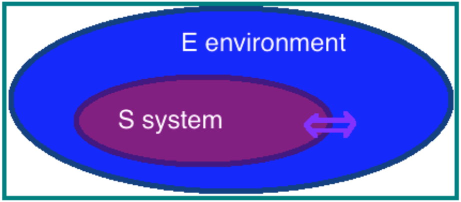

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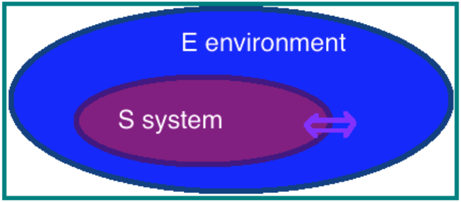

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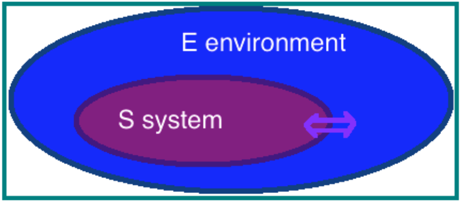

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The infinitesimal generator $\mathcal{L}_\Lambda^*$ of the previous semigroup of quantum channels is usually called **Liouvillian**, or **Lindbladian**.

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Quantum dissipative engineering,

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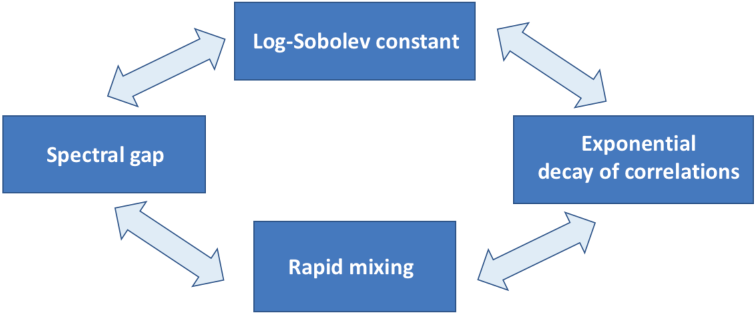

Log-Sobolev constant

Spectral gap

Exponential decay of correlations

Rapid mixing

QUANTUM SPIN SYSTEMS

$$\begin{tikzcd}
\mathrm{Ric}\,(\mathcal{L}) \geq \kappa \arrow[r, Rightarrow] & \mathrm{HWI}\,(\kappa) \arrow[r, Rightarrow, bend left, "\alpha=\kappa>0"] \arrow[r, Rightarrow, bend right, swap, "{W_{2,\mathcal{L}}<D,\ \alpha\propto D^{-2},\ (\Lambda_t)_{t\geq 0}\ \text{unital}}"] \arrow[rrr, Rightarrow, bend left, "{W_{2,\mathcal{L}}\leq D,\ \lambda\propto D^{-2}}"] & \mathrm{MLSI}(\alpha) \arrow[r, Rightarrow, "c_2=\alpha^{-1}"] & \mathrm{TC}_2(c_2) \arrow[r, Rightarrow, swap, "\lambda\propto c_2"] \arrow[d, Rightarrow, swap, "c_1=d\,c_2"] & \mathrm{PI}(\lambda) \arrow[r, Rightarrow] & \text{Exp.} \\
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Kastoryano-Temme, 2013

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Log-Sobolev constant ⇒ Rapid mixing.

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$$\left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \sqrt{2D(\rho_\Lambda || \sigma_\Lambda)}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t} \leq \sqrt{2\log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t}.$$

Log-Sobolev constant $\Rightarrow$ Rapid mixing.

PROBLEM

Find positive log-Sobolev constants!

Log-Sobolev constant

Log-Sobolev constant

The **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ is defined as:

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

If $\alpha(\mathcal{L}_\Lambda^*) > 0$:

$$D(\rho_t || \sigma_\Lambda) \leq D(\rho_\Lambda || \sigma_\Lambda) e^{-2\, \alpha(\mathcal{L}_\Lambda^*)\, t},$$

and with **Pinsker's inequality**, we have:

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Log-Sobolev constant ⇒ Rapid mixing.

Problem

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Main problem of my thesis

Develop a strategy to find positive log Sobolev constants.

Concrete problem

Provide sufficient static conditions on a Gibbs state which imply the existence of a positive log-Sobolev constant.

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Classical spin systems

(Cesi, Dai Pra-Paganoni-Posta, '02)

(1) Quasi-factorization of the entropy (in terms of a conditional entropy).

+

(2) Recursive geometric argument.
Lower bound for the global log-Sobolev constant in terms of the log-Sobolev constant of a size-fixed region.

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Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ. We define the **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ by

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Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ, $A \subseteq \Lambda$. We define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ on A by

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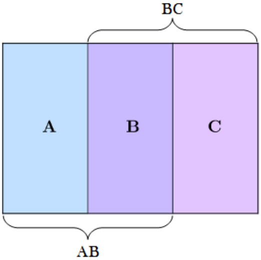
2. Quasi-factorization of the relative entropy

STATEMENT OF THE PROBLEM



PROBLEM

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in S_{ABC}$. Can we prove something like

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right] \text{ ?}$$

QUANTUM RELATIVE ENTROPY

$$D(\rho||\sigma) = \operatorname{tr}\left[\rho(\log \rho - \log \sigma)\right]$$

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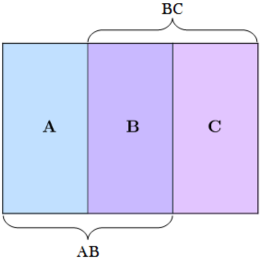

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Problem

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right]$$

Classical case, Dai Pra et al. '02

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{1 - 4\|h-1\|_\infty} \, \mu \left[\mathrm{Ent}_\mu(f \mid \mathcal{F}_1) + \mathrm{Ent}_\mu(f \mid \mathcal{F}_2) \right],$$

where $h = \dfrac{d\mu}{d\bar{\mu}}$.

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Entropy:

$$\operatorname{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Conditional entropy:

$$\operatorname{Ent}_\mu(f \mid \mathcal{G}) = \mu(f \log f \mid \mathcal{G}) - \mu(f \mid \mathcal{G}) \log \mu(f \mid \mathcal{G}).$$

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RELATIVE ENTROPY

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Let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$. The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

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PROPERTIES OF THE RELATIVE ENTROPY

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** $\rho_{AB} \mapsto D(\rho_{AB}||\sigma_{AB})$ is continuous.
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CHARACTERIZATION OF THE RE, Wilming et al. '17, Matsumoto '10

If $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$ satisfies $1 - 4$, then f is the relative entropy.

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CONDITIONAL RELATIVE ENTROPY

CONDITIONAL RELATIVE ENTROPY

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a **conditional relative entropy** in A as a function

$$D_A(\cdot||\cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and
   (2.1) $D_A(\rho_{AB}||\sigma_{AB}){=}0$ if, and only if, $\rho_{AB} = \sigma_{AB}^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma_{AB}^{1/2}$.
3. **Semi-superadditivity:** $D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$ and
   (3.1) **Semi-additivity:** if $\rho_{AB} = \rho_A \otimes \rho_B$,
   $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.
4. **Semi-motonicity:** For every quantum channel $\mathcal{T}$,

$$\begin{aligned} &D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) + D_B((\operatorname{tr}_A \circ \mathcal{T})(\rho_{AB})||(\operatorname{tr}_A \circ \mathcal{T})(\sigma_{AB})) \\ &\leq D_A(\rho_{AB}||\sigma_{AB}) + D_B(\operatorname{tr}_A(\rho_{AB})||\operatorname{tr}_A(\sigma_{AB})). \end{aligned}$$

REMARK

Consider for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$

$$D^{+}_{A,B}(\rho_{AB}||\sigma_{AB}) = D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}).$$

Then, $D^{+}_{A,B}$ verifies the following properties:

1. **Continuity:** $\rho_{AB} \mapsto D^{+}_{A,B}(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity:** $D^{+}_{A,B}(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity:** $D^{+}_{A,B}(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

However, it does not satisfy the property of monotonicity.

AXIOMATIC CHARACTERIZATION OF THE CRE (C-Lucia-Pérez García, '18)

The only possible conditional relative entropy is given by:

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B)$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

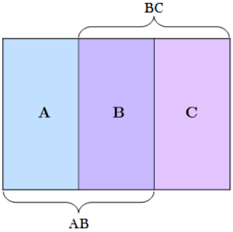

Figure: Choice of indices in $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Result of **quasi-factorization** of the relative entropy, for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$:

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

QUASI-FACTORIZATION FOR THE CRE (C-Lucia-Pérez García, '18)

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \frac{1}{1 - 2\|H(\sigma_{AC})\|_\infty} \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \, \sigma_{AC} \, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Note that $H(\sigma_{AC}) = 0$ if σ_{AC} is a tensor product between A and C.

$$\begin{aligned}(1-2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \leq \\ D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) = \\ = 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).\end{aligned}$$

$$\Leftrightarrow$$

$$(1+2\|H(\sigma_{AC})\|_\infty)D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

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This result is equivalent to:

$$\boxed{(1+2\|H(\sigma_{AB})\|_{\infty})D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

- **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.

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Due to:

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we have

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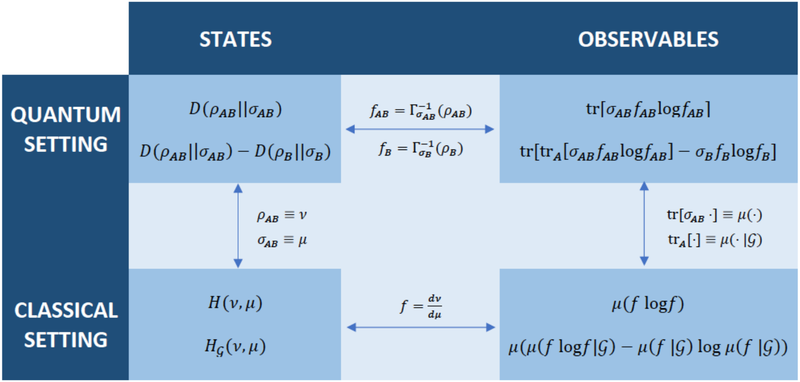
Relation with the classical case



	STATES		**OBSERVABLES**
QUANTUM SETTING	$D(\rho_{AB}\|\|\sigma_{AB})$ $D(\rho_{AB}\|\|\sigma_{AB}) - D(\rho_B\|\|\sigma_B)$	$f_{AB} = \Gamma^{-1}_{\sigma_{AB}}(\rho_{AB})$ ↔ $f_B = \Gamma^{-1}_{\sigma_B}(\rho_B)$	$\operatorname{tr}[\sigma_{AB} f_{AB} \log f_{AB}]$ $\operatorname{tr}[\operatorname{tr}_A[\sigma_{AB} f_{AB} \log f_{AB}] - \sigma_B f_B \log f_B]$
	↕ $\rho_{AB} \equiv \nu$ $\sigma_{AB} \equiv \mu$		↕ $\operatorname{tr}[\sigma_{AB}\,\cdot] \equiv \mu(\cdot)$ $\operatorname{tr}_A[\cdot] \equiv \mu(\cdot\,\|\mathcal{G})$
CLASSICAL SETTING	$H(\nu, \mu)$ $H_{\mathcal{G}}(\nu, \mu)$	$f = \frac{d\nu}{d\mu}$ ↔	$\mu(f \log f)$ $\mu(\mu(f \log f\|\mathcal{G}) - \mu(f\|\mathcal{G}) \log \mu(f\|\mathcal{G}))$

Figure: Identification between classical and quantum quantities when the states considered are classical.

3. Log-Sobolev constant

Quantum spin lattices

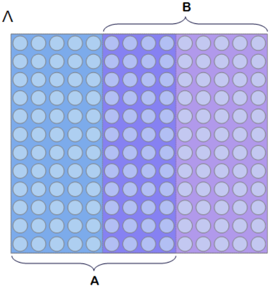


Figure: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

Problem

For a certain $\mathcal{L}_\Lambda^*$, can we prove $\alpha(\mathcal{L}_\Lambda^*) > 0$ using the result of quasi-factorization of the relative entropy?

Theorem (C-Lucia-Pérez García, '18)

The **heat-bath dynamics**, with product fixed point, has a positive log-Sobolev constant.

Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x \in \Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x \in \Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

THEOREM (C-Lucia-Pérez García, '18)

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$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \ \mathcal{L}_\Lambda^* = \sum_{x\in\Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

$$\mathcal{L}_\Lambda^*(\rho_\Lambda) = \sum_{x\in\Lambda} (\sigma_x \otimes \rho_{x^c} - \rho_\Lambda).$$

GENERAL QUASI-FACTORIZATION FOR σ A TENSOR PRODUCT

Let $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$ and $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x\in\Lambda} \sigma_x$. The following inequality holds:

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x\in\Lambda} D_x(\rho_\Lambda || \sigma_\Lambda).$$

THEOREM (C-Lucia-Pérez García, '18)

The **heat-bath dynamics**, with product fixed point, has a positive log-Sobolev constant.

Consider the local and global Lindbladians

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LOG-SOBOLEV CONSTANT

CONDITIONAL LOG-SOBOLEV CONSTANT

For $x \in \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in x by

$$\alpha_\Lambda(\mathcal{L}_x^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_x(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and $D_x(\rho_\Lambda || \sigma_\Lambda)$ is the conditional relative entropy.

LEMMA

$$\alpha_\Lambda(\mathcal{L}_x^*) \geq \frac{1}{2}.$$

$$\begin{aligned}
D(\rho_\Lambda || \sigma_\Lambda) &\leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \\
&\leq \sum_{x \in \Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2\alpha_\Lambda(\mathcal{L}_x^*)} \\
&\leq \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x \in \Lambda} - \operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\
&= \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]\right) \\
&\leq \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]\right).
\end{aligned}$$

Positive log-Sobolev constant

$$\alpha(\mathcal{L}_\Lambda^*) \geq \frac{1}{2}.$$

EXAMPLES OF POSITIVE LOG-SOBOLEV CONSTANTS

ASSUMPTION 1

In a tripartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_C \otimes \mathcal{H}_B$, A and B not connected, we have

$$\left\| h(\sigma_{AB}) \right\|_\infty = \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \sigma_{AB} \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty \leq K < \frac{1}{2}.$$

In particular, classical Gibbs states satisfy this.

ASSUMPTION 2

For any $B \subset \Lambda$, $B = B_1 \cup B_2$, it holds:

$$D_B(\rho_\Lambda || \sigma_\Lambda) \leq f(\sigma_{B\partial}) \left(D_{B_1}(\rho_\Lambda || \sigma_\Lambda) + D_{B_2}(\rho_\Lambda || \sigma_\Lambda) \right).$$

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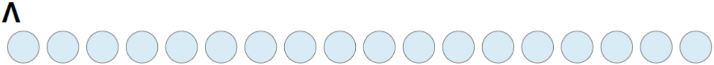
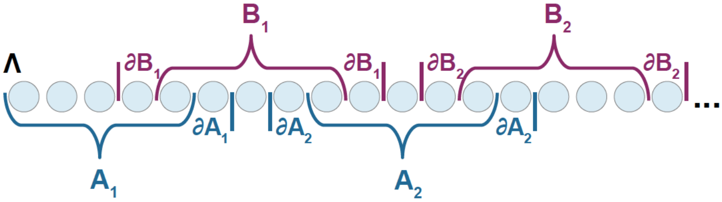
Sketch of the proof



Sketch of the proof

STEP 1



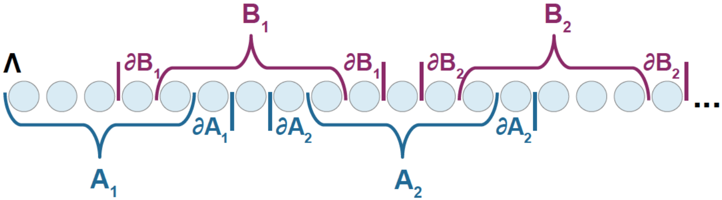
$$A = \bigcup_{i=1}^{n} A_i \text{ and } B = \bigcup_{j=1}^{n} B_j$$

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \frac{1}{1 - 2 \| h(\sigma_{A^c B^c}) \|_\infty} \left[D_A(\rho_\Lambda || \sigma_\Lambda) + D_B(\rho_\Lambda || \sigma_\Lambda) \right],$$

$$h(\sigma_{A^c B^c}) := \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} \sigma_{A^c B^c} \sigma_{A^c}^{-1/2} \otimes \sigma_{B^c}^{-1/2} - \mathbb{1}_{A^c B^c}.$$

SKETCH OF THE PROOF

STEP 1



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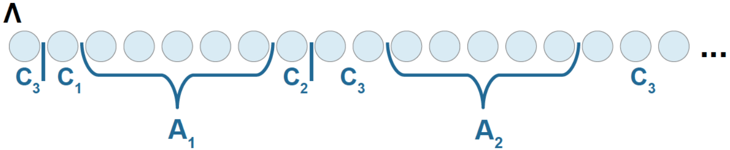
STEP 2

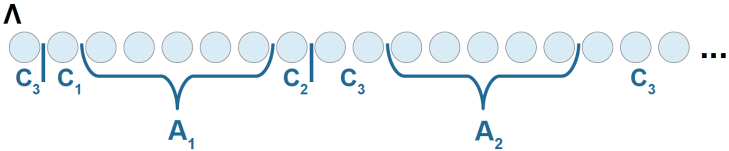


$$D_A(\rho_\Lambda||\sigma_\Lambda) \leq \sum_{i=1}^{n} D_{A_i}(\rho_\Lambda||\sigma_\Lambda)$$

σ_Λ is a QMC between $A_1 \leftrightarrow \partial A_1 \leftrightarrow \Lambda \setminus (A_1 \cup \partial A_1)$

$$\sigma_\Lambda = \bigoplus_{i\in I} \sigma_{A_1 (\partial a_1)_i^L} \otimes \sigma_{(\partial a_1)_i^R \Lambda \setminus (A_1 \cup \partial A_1)}$$

STEP 2



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STEP 3

$$\text{Assumption 1} \Rightarrow \alpha(\mathcal{L}_\Lambda^*) \geq \tilde{K} \min_{i \in \{1,\ldots n\}} \left\{ \alpha_\Lambda(\mathcal{L}_{A_i}^*), \alpha_\Lambda(\mathcal{L}_{B_i}^*) \right\}$$

STEP 4

$$\text{Assumption 2} \Rightarrow \alpha_\Lambda(\mathcal{L}_{A_i}^*) \geq g(\sigma_{A_i \partial}) > 0.$$

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$$\text{Assumption 2} \Rightarrow \alpha_\Lambda(\mathcal{L}_{A_i}^*) \geq g(\sigma_{A_i \partial}) > 0.$$

OPEN PROBLEMS

PROBLEM 1

Can we use any of the quasi-factorization results to prove log-Sobolev constants in a more general setting?

PROBLEM 2

Does this hold for greater dimension?

PROBLEM 3

Is there a better definition for conditional relative entropy?


рахмат
danke
謝謝
spas
ngiyabonga
شكراً جزيلاً
Баярлалаа
mersi
kia ora
barka
welalin
tack
teşekkür ederim
mahalo
tapadh leat
спасибо
faafetai lava
vinaka
спасибі
blagodaram
dank je
misaotra
matondo
paldies
grazzi
хвала
kiitos
dankie
nanni
nandri
dhanyavad
thank you
gracias
asante
manana
obrigada
tenki
enkosi
bedankt
bayarlalaa
gràcie
hvala
maururu
köszönöm
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didi madloba
sağolun
sukriya
kop khun krap
ありがとう
tanemirt
rahmet
grazie
arigatô
takk
dakujem
trugarez
najis tuke
kam sah hamnida
rahmat
terima kasih
ευχαριστώ
diolch
dhanyavadagalu
shukriya
mercé
мерси
তোমাকে ধন্যবাদ
감사합니다
xièxie
merci