Quantum conditional relative entropy and quasi-factorization of the relative entropy

Ángela Capel (ICMAT-UAM, Madrid)

Joint work with Angelo Lucia (U. Copenhagen and Caltech) and David Pérez-García (U. Complutense de Madrid).

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Contents

Motivation

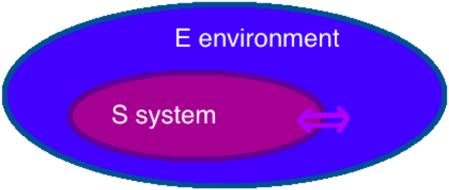

Figure: An open quantum many-body system.

- Interesting for information processing $\Rightarrow$ Open (unavoidable interactions).
- Dynamics of S is dissipative!
- The continuous-time evolution of a state on S is given by a **quantum Markov semigroup**.

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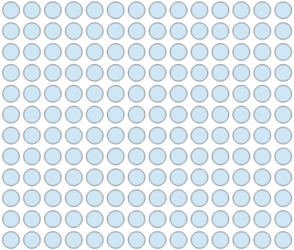

Figure: A quantum spin lattice system.

- Lattice $\Lambda \subset\subset \mathbb{Z}^d$.
- To every site $x \in \Lambda$ we associate $\mathcal{H}_x$ $(= \mathbb{C}^D)$.
- The global Hilbert space associated to Λ is $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathcal{H}_x$.

Dissipative quantum systems

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A **dissipative quantum system** is a 1-parameter continuous semigroup $\{\mathcal{T}_t^*\}_{t\geq 0}$ of completely positive, trace preserving (CPTP) maps (a.k.a. quantum channels) in $\mathcal{S}_\Lambda$.

$$\rho_\Lambda \xrightarrow{t} \rho_t := \mathcal{T}_t^*(\rho_\Lambda) = e^{t\mathcal{L}_\Lambda^*}(\rho_\Lambda) \xrightarrow{t\to\infty} \sigma_\Lambda$$

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Rapid mixing

We say that $\mathcal{L}_\Lambda^*$ satisfies **rapid mixing** if

$$\sup_{\rho_\Lambda \in \mathcal{S}_\Lambda} \left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \text{poly}(|\Lambda|) e^{-\gamma t}.$$

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Problem

Find examples of rapid mixing!

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Let $\mathcal{L}_\Lambda^* : \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ be a primitive reversible Lindbladian with stationary state σ_Λ. We define the **log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ by

$$\alpha(\mathcal{L}_\Lambda^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D(\rho_\Lambda || \sigma_\Lambda)}$$

If $\alpha(\mathcal{L}_\Lambda^*) > 0$:

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$$\left\| \rho_t - \sigma_\Lambda \right\|_1 \leq \sqrt{2D(\rho_\Lambda || \sigma_\Lambda)}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t} \leq \sqrt{2 \log(1/\sigma_{\min})}\, e^{-\alpha(\mathcal{L}_\Lambda^*)\,t}.$$

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Classical case

Classical entropy and conditional entropy

Probability space $(\Omega, \mathcal{F}, \mu)$. We define, for every $f > 0$, the **entropy** of f by

$$\mathrm{Ent}_\mu(f) = \mu(f \log f) - \mu(f) \log \mu(f).$$

Given a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, we define the **conditional entropy** of f in $\mathcal{G}$ by

$$\mathrm{Ent}_\mu(f \mid \mathcal{G}) = \mu(f \log f \mid \mathcal{G}) - \mu(f \mid \mathcal{G}) \log \mu(f \mid \mathcal{G}).$$

Lemma, Dai Pra et al. '02

$\mathcal{F}_1, \mathcal{F}_2$ sub-σ-algebras of $\mathcal{F}$:

$$\mathrm{Ent}_\mu(f) \leq \frac{1}{1 - 4\|h-1\|_\infty} \, \mu\left[\mathrm{Ent}_\mu(f \mid \mathcal{F}_1) + \mathrm{Ent}_\mu(f \mid \mathcal{F}_2)\right],$$

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Problem

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in S_{ABC}$. Can we prove something like

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC})\left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC})\right] \text{ ?}$$

Yes! (We will see how later)

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Relative entropy

Quantum relative entropy

Let $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$. The **quantum relative entropy** of ρ_Λ and σ_Λ is defined by:

$$D(\rho_\Lambda || \sigma_\Lambda) = \operatorname{tr}\left[\rho_\Lambda (\log \rho_\Lambda - \log \sigma_\Lambda)\right].$$

Properties of the relative entropy

Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$. The following properties hold:

1. **Continuity.** $\rho_{AB} \mapsto D(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Additivity.** $D(\rho_A \otimes \rho_B || \sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
3. **Superadditivity.** $D(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)$.
4. **Monotonicity.** $D(\rho_{AB}||\sigma_{AB}) \geq D(T(\rho_{AB})||T(\sigma_{AB}))$ for every quantum channel T.

Characterization of the RE, Wilming et al. '17, Matsumoto '10

If $f : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$ satisfies $1 - 4$, then f is the relative entropy.

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Conditional relative entropy

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Let $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. We define a **conditional relative entropy** in A as a function

$$D_A(\cdot||\cdot) : \mathcal{S}_{AB} \times \mathcal{S}_{AB} \to \mathbb{R}_0^+$$

verifying the following properties for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$:

1. **Continuity:** The map $\rho_{AB} \mapsto D_A(\rho_{AB}||\sigma_{AB})$ is continuous.
2. **Non-negativity:** $D_A(\rho_{AB}||\sigma_{AB}) \geq 0$ and
   (2.1) $D_A(\rho_{AB}||\sigma_{AB})=0$ if, and only if, $\rho_{AB} = \sigma_{AB}^{1/2}\sigma_B^{-1/2}\rho_B\sigma_B^{-1/2}\sigma_{AB}^{1/2}$.
3. **Semi-superadditivity:** $D_A(\rho_{AB}||\sigma_A \otimes \sigma_B) \geq D(\rho_A||\sigma_A)$ and
   (3.1) **Semi-additivity:** if $\rho_{AB} = \rho_A \otimes \rho_B$,
   $D_A(\rho_A \otimes \rho_B||\sigma_A \otimes \sigma_B) = D(\rho_A||\sigma_A)$.
4. **Semi-monotonicity:** For every quantum channel $\mathcal{T}$,

$$D_A(\mathcal{T}(\rho_{AB})||\mathcal{T}(\sigma_{AB})) + D_B((\operatorname{tr}_A \circ \mathcal{T})(\rho_{AB})||(\operatorname{tr}_A \circ \mathcal{T})(\sigma_{AB}))$$
$$\leq D_A(\rho_{AB}||\sigma_{AB}) + D_B(\operatorname{tr}_A(\rho_{AB})||\operatorname{tr}_A(\sigma_{AB})).$$

Remark

Consider for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$

$$D^+_{A,B}(\rho_{AB}||\sigma_{AB}) = D_A(\rho_{AB}||\sigma_{AB}) + D_B(\rho_{AB}||\sigma_{AB}).$$

Then, $D^+_{A,B}$ verifies the following properties:

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However, it does not satisfy the property of monotonicity.

Axiomatic characterization of the conditional relative entropy

The only possible conditional relative entropy is given by:

$$D_A(\rho_{AB}||\sigma_{AB}) = D(\rho_{AB}||\sigma_{AB}) - D(\rho_B||\sigma_B)$$

for every $\rho_{AB}, \sigma_{AB} \in \mathcal{S}_{AB}$.

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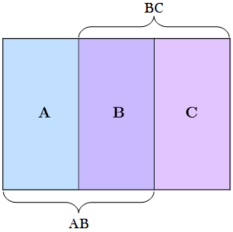

Figure: Choice of indices in $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Result of **quasi-factorization** of the relative entropy, for every $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$:

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \xi(\sigma_{ABC}) \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where $\xi(\sigma_{ABC})$ depends only on σ_{ABC} and measures how far σ_{AC} is from $\sigma_A \otimes \sigma_C$.

Quasi-factorization for the CRE

Let $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$. Then, the following inequality holds

$$D(\rho_{ABC}||\sigma_{ABC}) \leq \frac{1}{1-2\|H(\sigma_{AC})\|_\infty} \left[D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) \right],$$

where

$$H(\sigma_{AC}) = \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} \, \sigma_{AC} \, \sigma_A^{-1/2} \otimes \sigma_C^{-1/2} - \mathbb{1}_{AC}.$$

Note that $H(\sigma_{AC}) = 0$ if σ_{AC} is a tensor product between A and C.

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$$\text{Ent}_\mu(f) \leq \frac{1}{1-4\|h-1\|_\infty} \, \mu \left[\text{Ent}_\mu(f \mid \mathcal{F}_1) + \text{Ent}_\mu(f \mid \mathcal{F}_2) \right],$$

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$$
\begin{aligned}
(1 - 2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \leq \\
D_{AB}(\rho_{ABC}||\sigma_{ABC}) + D_{BC}(\rho_{ABC}||\sigma_{ABC}) = \\
= 2D(\rho_{ABC}||\sigma_{ABC}) - D(\rho_C||\sigma_C) - D(\rho_A||\sigma_A).
\end{aligned}
$$

$$\Leftrightarrow$$

$$(1 + 2\|H(\sigma_{AC})\|_\infty) D(\rho_{ABC}||\sigma_{ABC}) \geq D(\rho_A||\sigma_A) + D(\rho_C||\sigma_C).$$

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This result is equivalent to:

$$\boxed{(1+2\|H(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Recall:

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Relation with the classical case

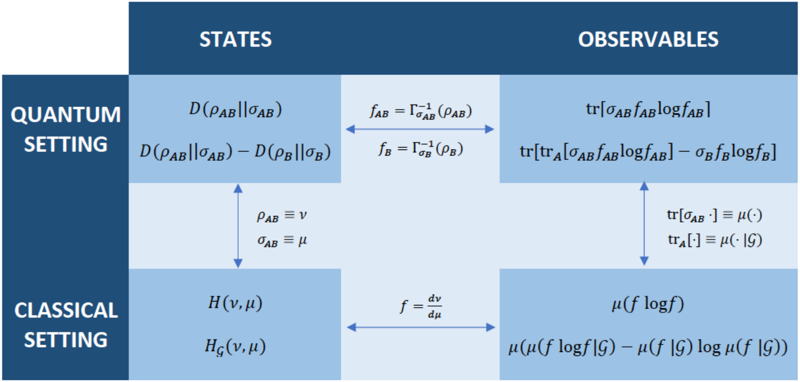

Figure: Identification between classical and quantum quantities when the states considered are classical.

Sketch of the proof of quasi-factorization

$$\boxed{(1+2\|H(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Step 1

$$D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B) - \log \operatorname{tr} M, \tag{1}$$

where $M = \exp\left[\log \sigma_{AB} - \log \sigma_A \otimes \sigma_B + \log \rho_A \otimes \rho_B\right]$.

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where $M = \exp\left[\log \sigma_{AB} - \log \sigma_A \otimes \sigma_B + \log \rho_A \otimes \rho_B\right]$.

Step 2

$$\log \operatorname{tr} M \leq \operatorname{tr}[L(\sigma_{AB})\,(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)], \tag{2}$$

where

$$L(\sigma_{AB}) = \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}.$$

Sketch of the proof of quasi-factorization

$$\boxed{(1+2\|H(\sigma_{AB})\|_\infty)D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B)}.$$

Step 1

$$D(\rho_{AB}||\sigma_{AB}) \geq D(\rho_A||\sigma_A) + D(\rho_B||\sigma_B) - \log \operatorname{tr} M, \tag{1}$$

where $M = \exp\left[\log \sigma_{AB} - \log \sigma_A \otimes \sigma_B + \log \rho_A \otimes \rho_B\right]$.

Step 2

$$\log \operatorname{tr} M \leq \operatorname{tr}[L(\sigma_{AB})\,(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)], \tag{2}$$

where

$$L(\sigma_{AB}) = \mathcal{T}_{\sigma_A \otimes \sigma_B}(\sigma_{AB}) - \mathbb{1}_{AB}.$$

Theorem (Lieb, '73)

Let g a positive operator, and define

$$\mathcal{T}_g(f) = \int_0^\infty \mathrm{d}t \, (g+t)^{-1} f (g+t)^{-1}.$$

$\mathcal{T}_g$ is positive-semidefinite if g is. We have that

$$\operatorname{tr}[\exp(-f+g+h)] \leq \operatorname{tr}\left[e^h \mathcal{T}_{e^f}(e^g)\right].$$

Lemma (Sutter et al., '17)

For $f \in \mathcal{S}_{AB}$ and $g \in \mathcal{A}_{AB}$ the following holds:

$$\mathcal{T}_g(f) = \int_{-\infty}^{\infty} dt \, \beta_0(t) \, g^{\frac{-1-it}{2}} \, f \, g^{\frac{-1+it}{2}},$$

with

$$\beta_0(t) = \frac{\pi}{2}(\cosh(\pi t)+1)^{-1}.$$

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Step 3

$$\operatorname{tr}[L(\sigma_{AB})\,(\rho_A - \sigma_A) \otimes (\rho_B - \sigma_B)] \leq 2\|L(\sigma_{AB})\|_\infty D(\rho_{AB}||\sigma_{AB}). \qquad (3)$$

Hölder's inequality + Tensorization of Schatten norms + Pinsker's inequality + Data Processing inequality.

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$$\|L(\sigma_{AB})\|_\infty \leq \left\| \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} \, \sigma_{AB} \, \sigma_A^{-1/2} \otimes \sigma_B^{-1/2} - \mathbb{1}_{AB} \right\|_\infty . \quad (4)$$

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Quantum spin lattices

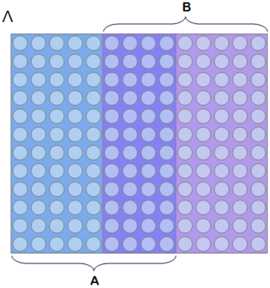

Figure: A quantum spin lattice system Λ and $A, B \subseteq \Lambda$ such that $A \cup B = \Lambda$.

Problem

For a certain $\mathcal{L}_\Lambda^*$, can we prove $\alpha(\mathcal{L}_\Lambda^*) > 0$ using the result of quasi-factorization of the relative entropy?

(1) Quasi-factorization of the relative entropy.

+

(2) Recursive geometric argument.
Lower bound for the log-Sobolev constant in terms of a conditional log-Sobolev constant.

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(3) Positive (and size-independent) conditional log-Sobolev constant.

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$\Downarrow$

Positive log-Sobolev constant.

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Positive log-Sobolev constant.

General quasi-factorization for σ a tensor product

Let $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{H}_x$ and $\rho_\Lambda, \sigma_\Lambda \in \mathcal{S}_\Lambda$ such that $\sigma_\Lambda = \bigotimes_{x \in \Lambda} \sigma_x$. The following inequality holds:

$$D(\rho_\Lambda || \sigma_\Lambda) \leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda). \tag{5}$$

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Consider the local and global Lindbladians

$$\mathcal{L}_x^* := \mathbb{E}_x^* - \mathbb{1}_\Lambda, \; \mathcal{L}_\Lambda^* = \sum_{x\in\Lambda} \mathcal{L}_x^*$$

Since

$$\mathbb{E}_x^*(\rho_\Lambda) = \sigma_\Lambda^{1/2} \sigma_{x^c}^{-1/2} \rho_{x^c} \sigma_{x^c}^{-1/2} \sigma_\Lambda^{1/2} = \sigma_x \otimes \rho_{x^c}$$

for every $\rho_\Lambda \in \mathcal{S}_\Lambda$, we have

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Conditional log-Sobolev constant

For $x \in \Lambda$, we define the **conditional log-Sobolev constant** of $\mathcal{L}_\Lambda^*$ in x by

$$\alpha_\Lambda(\mathcal{L}_x^*) := \inf_{\rho_\Lambda \in \mathcal{S}_\Lambda} \frac{-\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2D_x(\rho_\Lambda || \sigma_\Lambda)},$$

where σ_Λ is the fixed point of the evolution, and $D_x(\rho_\Lambda||\sigma_\Lambda)$ is the conditional relative entropy.

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Lemma

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\begin{aligned}
D(\rho_\Lambda || \sigma_\Lambda) &\leq \sum_{x \in \Lambda} D_x(\rho_\Lambda || \sigma_\Lambda) \\
&\leq \sum_{x \in \Lambda} \frac{- \operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]}{2 \alpha_\Lambda(\mathcal{L}_x^*)} \\
&\leq \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x \in \Lambda} - \operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\
&= \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \right) \\
&\leq \left(- \operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \right). \qquad (2)
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&\leq \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \sum_{x \in \Lambda} -\operatorname{tr}[\mathcal{L}_x^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)] \\
&= \frac{1}{2 \inf\limits_{x \in \Lambda} \alpha_\Lambda(\mathcal{L}_x^*)} \left(-\operatorname{tr}[\mathcal{L}_\Lambda^*(\rho_\Lambda)(\log \rho_\Lambda - \log \sigma_\Lambda)]\right) \\
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Open problems

Problem 1

Can we use any of the quasi-factorization results to prove log-Sobolev constants in a more general setting?

(Kastoryano-Brandao, '15) The heat-bath dynamics, with σ_Λ the Gibbs state of a commuting Hamiltonian, has positive spectral gap. $\Rightarrow$ Log-Sobolev constant?

Problem 2

Is there a better definition for conditional relative entropy?

For further knowledge, Arxiv: 1705.03521 and 1804.09525