A strengthened data processing inequality for the Belavkin-Staszewski relative entropy

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Joint work with Andreas Bluhm (U. Copenhagen)

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Main concepts

Relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **relative entropy** is defined as:

$$D(\sigma||\rho) := \operatorname{tr}[\sigma(\log \sigma - \log \rho)].$$

Belavkin-Staszewski relative entropy

Given $\sigma > 0, \rho > 0$ states on a matrix algebra $\mathcal{M}$, their **BS-entropy** is defined as:

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The following holds for every $\sigma > 0, \rho > 0$:

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Data processing inequality

Quantum channel: $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ CPTP map.

- $\sigma > 0 \mapsto \mathcal{T}(\sigma) > 0$.
- $\mathcal{T} \otimes \mathrm{Id}_n : \mathcal{M} \otimes \mathcal{M}_n \to \mathcal{M} \otimes \mathcal{M}_n$ is positive for every $n \in \mathbb{N}$.
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Conditions for equality, Petz '86

$D(\sigma||\rho) = D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \Leftrightarrow \sigma = \mathcal{P}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma)$, for $\mathcal{P}_{\mathcal{T}}^{\rho}$ a recovery map.

Petz recovery map: $\mathcal{R}_{\mathcal{T}}^{\rho}(\cdot) := \rho^{1/2}\mathcal{T}^* \left(\mathcal{T}(\rho)^{-1/2}(\cdot)\mathcal{T}(\rho)^{-1/2} \right) \rho^{1/2}$.

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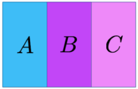

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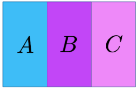

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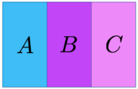

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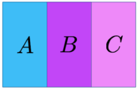

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Strengthened bounds for DPI of RE

Operational meaning of $D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho))$

- **Thermodynamics:** Cost of a certain quantum process (Faist et al, '18).
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CMI: $I(A:C|B)_\sigma = D(\sigma_{ABC}||\rho_{ABC}) - D(\sigma_{BC}||\rho_{BC}).$

$$I(A:C|B)_\sigma \geq \inf_{\eta_{ABC} \text{recov.}} \left(-2\log_2 F(\sigma_{ABC}, \eta_{ABC})\right),$$

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$$\mathcal{V}_{BC} \circ \mathcal{R}^{\sigma_{BC}}_{\mathrm{tr}_C} \circ \mathcal{U}_B(\sigma_{AB}) = V_{BC}\sigma_{BC}^{1/2}\sigma_B^{-1/2}U_B\sigma_{AB}U_B^*\sigma_B^{-1/2}\sigma_{BC}^{1/2}V_{BC}^*,$$

we have

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Extensions and improvements of the previous result:
$D(\sigma||\rho) - D(\mathcal{T}(\sigma)||\mathcal{T}(\rho)) \geq (1), (2), (3)$, where:

$$(1) := -\int \beta_0(t) \log F\left(\sigma, \mathcal{R}_{\mathcal{T}}^{\rho,[t]} \circ \mathcal{T}(\sigma)\right) \mathrm{d}t \textbf{ (Junge et al. '15)},$$

with

$$\mathcal{R}_{\mathcal{T}}^{\rho,[t]}(\cdot) = \rho^{\frac{1+it}{2}} \mathcal{T}^* \left(\mathcal{T}(\rho)^{\frac{-1-it}{2}}(\cdot)\mathcal{T}(\rho)^{\frac{-1+it}{2}}\right) \rho^{\frac{1-it}{2}}$$

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$$(2) := D_M \left(\sigma \middle\| \int \beta_0(t) \, \mathcal{R}_{\mathcal{T}}^{\sigma,[t]} \circ \mathcal{T}(\sigma)\right) \mathrm{d}t$$ **(Sutter-Berta-Tomamichel '16)**,

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Can we find a lower bound for the DPI in terms of $D(\sigma || \mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma))$*?*

Answer: It is not possible **(Brandao et al. '15, Fawzi^2 '17)**.

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$$D(\sigma_{ABC}||\mathcal{R}^{\sigma_{BC}}_{\text{tr}_C} \circ \text{tr}_C[\sigma_{ABC}]) + \Lambda_{\max}(\sigma_{AB}||\mathcal{R}_{B\to B}) \geq I(A:C|B)_\sigma,$$

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(Carlen-Vershynina '17) $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ conditional expectation, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$:

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(Carlen-Vershynina '18) Extension to standard f-divergences.

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Some definitions

Conditional expectation

Let $\mathcal{M}$ matrix algebra with matrix subalgebra $\mathcal{N}$. There exists a unique linear mapping $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ such that

1. $\mathcal{E}$ is a positive map,
2. $\mathcal{E}(B) = B$ for all $B \in \mathcal{N}$,
3. $\mathcal{E}(AB) = \mathcal{E}(A)B$ for all $A \in \mathcal{M}$ and all $B \in \mathcal{N}$,
4. $\mathcal{E}$ is trace preserving.

A map fulfilling (1)-(3) is called a *conditional expectation*.

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Operator convex

Let $\mathcal{I} \subseteq \mathbb{R}$ interval and $f : \mathcal{I} \to \mathbb{R}$. If

$$f(\lambda A + (1-\lambda)B) \leq \lambda f(A) + (1-\lambda) f(B)$$

for all Hermitian A, $B \in \mathcal{B}(\mathcal{H})$ with spectrum contained in $\mathcal{I}$, all $\lambda \in [0,1]$, and for all finite-dimensional Hilbert spaces $\mathcal{H}$, then f is *operator convex*.

Standard f-divergences

(Hiai-Mosonyi '17)

Standard f-divergences

Let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$S_f(\sigma \| \rho) = \operatorname{tr}\left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) \rho^{1/2}\right]$$

is the *standard f-divergence*.

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$$S_f(\sigma \| \rho) = \operatorname{tr}\left[\rho^{1/2} f(L_\sigma R_{\rho^{-1}}) \rho^{1/2}\right]$$

is the *standard f-divergence.*

Example: Let $f(x) = x \log x$. Then,

$$S_f(\sigma \| \rho) = \operatorname{tr}[\sigma(\log \sigma - \log \rho)]$$

defines the relative entropy $D(\sigma \| \rho)$.

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Standard f-divergences

Conditions for equality

Let $\sigma > 0$, $\rho > 0$ be on $\mathcal{M}$ and let $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a 2PTP linear map. Then, the following are equivalent:

1. There exists a TP map $\hat{\mathcal{T}} : \mathcal{B} \to \mathcal{M}$ such that $\hat{\mathcal{T}}(\mathcal{T}(\rho)) = \rho$ and $\hat{\mathcal{T}}(\mathcal{T}(\sigma)) = \sigma$.
2. $S_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) = S_f(\sigma \| \rho)$ for all operator convex f on $[0, \infty)$.
3. $\mathcal{R}^{\rho}_{\mathcal{T}}(\mathcal{T}(\sigma)) = \sigma$.

Maximal f-divergences

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Let $f : (0, \infty) \to \mathbb{R}$ be an operator convex function and $\sigma > 0$, $\rho > 0$ be two states on a matrix algebra $\mathcal{M}$. Then,

$$\hat{S}_f(\sigma \| \rho) = \operatorname{tr}\left[\rho^{1/2} f(\rho^{-1/2} \sigma \rho^{-1/2}) \rho^{1/2}\right]$$

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Let $\sigma > 0$, $\rho > 0$ be on $\mathcal{M}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{B}$ be a PTP linear map. Then, the following are equivalent:

1. $\hat{S}_f(\mathcal{T}(\sigma) \| \mathcal{T}(\rho)) = \hat{S}_f(\sigma \| \rho)$ for all operator convex functions on $[0, \infty)$.
2. $\operatorname{tr}\left[\mathcal{T}(\sigma)^2 \mathcal{T}(\rho)^{-1}\right] = \operatorname{tr}\left[\sigma^2 \rho^{-1}\right]$.

Relation between f-divergences

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For every two states $\sigma > 0$, $\rho > 0$ on $\mathcal{M}$ and every operator convex function $f : (0, \infty) \to \mathbb{R}$,

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For maximal f-divergences, there is no equivalent condition for equality in DPI which provides a explicit expression of recovery for σ.

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Questions

BS recovery condition

Can we prove an equivalent condition for equality in DPI for the BS entropy (or for maximal f-divergences) which provides a explicit expression of recovery for σ?

Strengthened DPI for BS entropy

Following Carlen-Vershynina, can we provide a lower bound for the DPI for the BS entropy (or for maximal f-divergences) in terms of a (hypothetical) BS recovery condition?

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Equivalent conditions for equality on DPI

$\Gamma := \sigma^{-1/2}\rho\sigma^{-1/2}$ and $\Gamma_{\mathcal{T}} := \sigma_{\mathcal{T}}^{-1/2}\rho_{\mathcal{T}}\sigma_{\mathcal{T}}^{-1/2}$
$\rho_{\mathcal{T}} := \mathcal{T}(\rho)$, $\sigma_{\mathcal{T}} := \mathcal{T}(\sigma)$

Equivalent conditions for equality on DPI (Bluhm-C. '19)

Let $\mathcal{M}$ and $\mathcal{N}$ be matrix algebras, $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ a quantum channel, $\sigma > 0$, $\rho > 0$ two quantum states on $\mathcal{M}$. The following are equivalent:

1. $D_{\text{BS}}(\sigma\|\rho) = D_{\text{BS}}(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}})$.
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$$\mathcal{B}_{\mathcal{T}}^{\sigma}(\cdot) := \sigma\mathcal{T}^*(\mathcal{T}(\sigma)^{-1}(\cdot)).$$

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Consequences

Note: Although they can be seen as a consequence of the previous result, the following facts were previously known.

Corollary

$$\begin{aligned} D_{\mathrm{BS}}(\sigma\|\rho) = D_{\mathrm{BS}}(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}) &\Leftrightarrow \rho = \mathcal{B}_{\mathcal{T}}^{\sigma} \circ \mathcal{T}(\rho) \\ &\Leftrightarrow \sigma = \mathcal{B}_{\mathcal{T}}^{\rho} \circ \mathcal{T}(\sigma) \\ &\Leftrightarrow D_{\mathrm{BS}}(\rho\|\sigma) = D_{\mathrm{BS}}(\rho_{\mathcal{T}}\|\sigma_{\mathcal{T}}). \end{aligned}$$

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$$D(\sigma\|\rho) = D(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}) \implies D_{BS}(\sigma\|\rho) = D_{BS}(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}).$$

Equivalently,

$$\sigma = \mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma) \implies \sigma = \mathcal{B}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma).$$

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Strengthened DPI for the BS-entropy

Strengthened DPI for the BS-entropy (Bluhm-C. '19)

Let $\mathcal{M}$ and $\mathcal{N}$ be matrix algebras, $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ a quantum channel, $\sigma > 0$, $\rho > 0$ two quantum states on $\mathcal{M}$. Then,

$$D_{\mathrm{BS}}(\sigma\|\rho) - D_{\mathrm{BS}}(\sigma_{\mathcal{T}}\|\rho_{\mathcal{T}}) \geq \left(\frac{\pi}{8}\right)^4 \|\Gamma\|_\infty^{-4} \left\|\sigma^{-1}\right\|_\infty^{-2} \left\|\rho - \sigma\mathcal{T}^*\left(\sigma_{\mathcal{T}}^{-1}\rho_{\mathcal{T}}\right)\right\|_2^4 .$$

Step 1

For $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ a conditional expectation, $\sigma_{\mathcal{N}} := \mathcal{E}(\sigma)$ and $\rho_{\mathcal{N}} := \mathcal{E}(\rho)$:

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Strengthened DPI for the BS-entropy

Steps 1 and 2

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Step 3

For quantum channels $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ (Stinespring's dilation):

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Strengthened DPI for maximal f-divergences

Strengthened DPI for maximal f-divergences (Bluhm-C. '19)

Let $\mathcal{M}$ and $\mathcal{N}$ be matrix algebras, $\mathcal{T} : \mathcal{M} \to \mathcal{N}$ a quantum channel, $\sigma > 0$, $\rho > 0$ two quantum states on $\mathcal{M}$ and $f : (0, \infty) \to \mathbb{R}$ an operator convex function with transpose $\tilde{f}$. We assume that $\tilde{f}$ is operator monotone decreasing and such that $\mu_{-\tilde{f}}$ is absolutely continuous with respect to Lebesgue measure. Moreover, we assume that for every $T \geq 1$, there exist constants $\alpha \geq 0$, $C > 0$ satisfying $\mathrm{d}\mu_{-\tilde{f}}(t)/\mathrm{d}t \geq (CT^{2\alpha})^{-1}$ for all $t \in [1/T, T]$ and such that

$$\left(\frac{(2\alpha+1)\sqrt{C}}{4} \frac{(\hat{S}_f(\sigma\|\rho) - \hat{S}_f(\sigma_\mathcal{T}\|\rho_\mathcal{T}))^{1/2}}{1 + \|\Gamma\|_\infty} \right)^{\frac{1}{1+\alpha}} \leq 1.$$

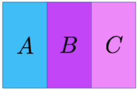
Then, there is a constant $L_\alpha > 0$ such that

$$\hat{S}_f(\sigma\|\rho) - \hat{S}_f(\sigma_\mathcal{T}\|\rho_\mathcal{T}) \geq$$
$$\geq \frac{L_\alpha}{C} \left(1 + \|\Gamma\|_\infty\right)^{-(4\alpha+2)} \|\Gamma\|_\infty^{-(2\alpha+2)} \left\|\sigma^{-1}\right\|_\infty^{-(2\alpha+2)} \left\|\rho - \sigma\mathcal{T}^*\left(\sigma_\mathcal{T}^{-1}\rho_\mathcal{T}\right)\right\|_2^{4(\alpha+1)}.$$

Comparison results for the relative entropy and the BS-entropy

Relative entropy	**BS-entropy**
$\mathrm{tr}[\sigma(\log\sigma - \log\rho)]$	$\mathrm{tr}\left[\sigma\log\left(\sigma^{1/2}\rho^{-1}\sigma^{1/2}\right)\right]$
$\rho = \rho^{1/2}\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1/2}\mathcal{T}(\sigma)\mathcal{T}(\rho)^{-1/2}\right)\rho^{1/2}$	$\sigma = \rho\,\mathcal{T}^*\left(\mathcal{T}(\rho)^{-1}\mathcal{T}(\sigma)\right)$
$\left(\frac{\pi}{8}\right)^4 \lVert L_\rho R_{\sigma^{-1}}\rVert_\infty^{-2} \lVert \mathcal{R}_{\mathcal{E}}^{\sigma}(\rho_{\mathcal{N}}) - \rho\rVert_1^4$	$\left(\frac{\pi}{8}\right)^4 \lVert\Gamma\rVert_\infty^{-4} \lVert\sigma^{-1}\rVert_\infty^{-2} \lVert\rho - \mathcal{B}_{\mathcal{T}}^{\sigma}\circ\mathcal{T}(\rho)\rVert_2^4$
Extension to standard f-divergences	Extension to maximal f-divergences

Future work



Particular case: $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Quantum channel: $\mathcal{T} = \operatorname{tr}_C$.

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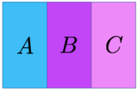
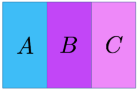


Particular case: $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Quantum channel: $\mathcal{T} = \operatorname{tr}_C$.

Consider $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$ such that $\rho_{ABC} = \frac{\mathbb{1}_A}{d_A} \otimes \sigma_{BC}$.

Future work



Particular case: $\mathcal{H}_{ABC} = \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$.

Quantum channel: $\mathcal{T} = \operatorname{tr}_C$.

Consider $\rho_{ABC}, \sigma_{ABC} \in \mathcal{S}_{ABC}$ such that $\rho_{ABC} = \frac{\mathbb{1}_A}{d_A} \otimes \sigma_{BC}$.

$$\sigma = \mathcal{R}^{\rho}_{\mathcal{T}} \circ \mathcal{T}(\sigma) \rightsquigarrow \sigma_{ABC} = \sigma_{BC}^{1/2} \, \sigma_B^{-1/2} \, \sigma_{AB} \, \sigma_B^{-1/2} \, \sigma_{BC}^{1/2}.$$

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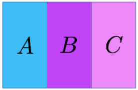

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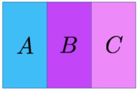

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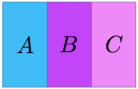

(Bluhm-C. '20)

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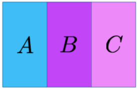
Define a **BS recoverable state** as a state $\sigma_{ABC} \in \mathcal{S}_{ABC}$ such that $\sigma_{ABC} = \sigma_{BC}\, \sigma_B^{-1}\, \sigma_{AB}$.

Future work



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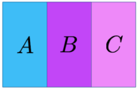
Question

Is the set of BS recoverable states robust?

Future work



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References

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thank you
danke
рахмат
謝謝
ngiyabonga
شكراً جزيلاً
teşekkür ederim
Баярлалаа
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